Cantor's Diagonal Argument: An Extension to the Socialist Calculation Debate

Robert P. Murphy

The socialist calculation debate is one of the most famous episodes in the history of the Austrian School. Provoked by Ludwig von Mises's (1990) original salvo in 1920,[1] the debate forced socialist theorists to refine their position. Before the calculation argument, opponents of socialism generally cited the issue of incentives; if workers are not paid in accordance with their productive contributions (as they are under capitalism), then we should expect widespread shirking. Those focusing on incentives also worried whether those exercising entrepreneurial functions would not exert themselves under socialism as much as they would under capitalism, without the lure of profits (and the punishment of losses). The rhetorical virtue of Mises's argument was that it had nothing to do with claims about human nature or empirical facts. On the contrary, Mises argued that even *in theory* socialism could not efficiently allocate productive resources, because (without market prices for the means of production) the central planners would honestly have no idea of the economic value of the factors at their disposal. There would thus be no analog of the profit and loss test to determine, even in retrospect, whether a given economic plan made an efficient use of society's scarce resources.[2]

In response to this position, Dickinson (1933) argued that rational allocation of productive resources could be achieved without private property, at

---

Robert Murphy is a visiting assistant professor of economics at Hillsdale College. He would like to thank Mark Watson for correcting a mistake in his summary of Cantor's argument.

[1]A note on citations: Mises's article appeared in German in 1920. An English translation, "Economic Calculation in the Socialist Commonwealth," appeared in Hayek's (1990) collection *Collectivist Economic Planning* in 1935.

[2]The current paper assumes a basic familiarity with the socialist calculation debate. An extensive (and excellent) analysis is Lavoie (1981).

THE QUARTERLY JOURNAL OF AUSTRIAN ECONOMICS VOL. 9, NO. 2 (SUMMER 2006): 3–11

least in theory.[3] Dickinson's scheme relied on a simultaneous system of equations such as the ones used in mathematical economics. Just as a Walrasian theorist could characterize an efficient use of resources equipped only with endowments, consumer preferences, and technology—the very items that Mises conceded to the hypothetical planners—so too (Dickinson claimed) the omnipotent dictator could, at least in principle, concoct a grand plan that channeled resources to their most desired ends.

Taking up the Austrian position, Hayek (1990) famously conceded that the "mathematical solution" of Dickinson and others "is not an impossibility in the sense that it is logically contradictory" (p. 207). Nonetheless, Hayek went on to argue, their schemes were still largely irrelevant in the debate over socialism, because

> what is practically relevant here is not the formal structure of the system, but the nature and amount of concrete information required if a numerical solution is to be attempted and the magnitude of the task which this numerical solution must involve in any modern community. (p. 208)

The socialist Oskar Lange quickly pounced (1936) on Hayek's qualified concession. After paying mock homage to Mises (and suggesting that the Central Planning Board of the socialist state should erect a statue of the Austrian economist in tribute to his contributions to socialist theory), Lange declared:

> Thus Professor Hayek and Professor Robbins [in their emphasis on the staggering number of equations necessary to actually implement the mathematical solution] have given up the essential point of Professor Mises' position and retreated to a second line of defence. On principle, they admit, the problem is soluble, but it is to be doubted whether in a socialist community it can be solved by a simple method of *trial and error*, as it is solved in the capitalist economy. (Lange 1936, p. 56, italics in original)

After his humorous tribute to Mises, Lange went on to propose his *tâtonnement* solution, in which central planners would tinker with the vector of official "prices" at which managers could exchange resources, until all such markets cleared.

In the present paper, I claim that the issue of the number of equations necessary for the so-called mathematical solution has not been given adequate attention, even by the Austrians. If the hypothetical planners are to actually use the Lange-Lerner approach to overcome all of the entrepreneurial incentive problems traditionally raised by critics of socialism, the vector of prices

[3]Specifically, Dickinson (1933) wrote, "It is the object of this article to refute the first of Mises's criticisms of socialism and to show that a rational pricing of instrumental goods is at least theoretically possible in a socialistic economy" (p. 238).

(that the Central Planning Board would announce to the citizens of the socialist commonwealth) would need to contain not merely billions or trillions of prices, but in fact an *uncountably infinite* number of them. If I can establish this proposition, then a standard result from set theory proves that the Lange-Lerner scheme is impossible *in principle*. Before defending these claims, I must first take a brief digression to explain Georg Cantor's famous "diagonal argument."

Cantor's Diagonal Argument

The mathematician Georg Cantor (1845–1918) developed a technique for comparing the relative sizes of different sets. Cantor proposed that two sets should be defined as having the same size (or technically, the same "cardinality") if the elements of one set could be placed in a one-to-one correspondence with the elements of the other set. This is straightforward for sets with a finite number of elements. For example, the set consisting of {apple, banana, pear} has the same cardinality as the set consisting of {dog, cat, goldfish} because we can associate each fruit with one pet, and when we do so, we know we won't "miss" any of the objects from either set.

Cantor then applied his technique to sets with an infinite number of elements, and he discovered some counterintuitive results. For example, Cantor realized that the set of all positive integers $\{1, 2, 3, \ldots, n, \ldots\}$ has the same cardinality as the set of all *even* positive integers, i.e., $\{2, 4, 6, \ldots, 2n, \ldots\}$. This may initially surprise the reader, because it seems as if there should be "twice as many" elements in the first set as in the second. However, such reasoning is dangerous when it comes to sets of infinite size. Using Cantor's technique, we must conclude that these two sets do indeed possess the same cardinality. This is because we can create a one-to-one mapping from each element of the first set to each element of the second set. For example, we can associate the "1" from the first set with the "2" from the second set, the "2" from the first set with the "4" from the second set, the "3" from the first set with the "6" from the second set, and so on *ad infinitum*. If we proceed in this fashion, we know we will eventually "catch" every element of every set; i.e., just as with the finite case, we know we won't "miss" any of the elements.

Our story does not stop here. Cantor then discovered that not all infinite sets have equal cardinality. That is, there are sets with an infinite number of elements that *cannot* be placed into a one-to-one correspondence with other sets that also possess an infinite number of elements. To prove this, Cantor devised an ingenious "diagonal argument," by which he demonstrated that the set of real numbers in the interval (0, 1) possessed a higher cardinality than the set of positive integers. A common way that mathematicians state this result (and one that is especially relevant to the socialist calculation debate) is that the real numbers cannot be *enumerated* (or *listed*).

Cantor's argument is a proof by contradiction. Suppose that we have come up with a one-to-one correspondence between the real numbers in the interval

(0, 1) and the set of positive integers. That means we could in principle list the elements of the former like this:

1. 0.792420349232...
2. 0.364934520293...
3. 0.943223202032...
4. 0.292557234203...
5. 0.512394395461...

:

:

Now if the cardinality of the two sets were really equal, we know that eventually (i.e., somewhere down this list of infinite length) we would write every single real number in between 0 and 1. Put alternatively, for any number in the interval (0, 1) that the reader picks, we know that it must be *somewhere* on this list, and that there must be some positive integer that corresponds to this particular real number.

But Cantor showed that the above procedure is vicious. He proved that there exists a real number that could not *possibly* be listed on the right hand side in the diagram above, no matter how far down we might check. We can construct such a number by making its first digit (after the decimal point) equal to 1 plus 7, because 7 is the first digit of the first number listed above. Then the second digit of the constructed number equals 1 plus 6, because 6 is the second digit of the second number listed above. The following table puts the relevant numbers in bold:

1. 0.**7**92420349232...
2. 0.3**6**4934520293...
3. 0.94**3**223202032...
4. 0.292**5**57234203...
5. 0.5123**9**4395461...

:

:

Adding one to each relevant digit, Cantor's constructed number would start off with the digits 0.87460 . . . (Note that if the relevant digit on the original list is a 9, then the corresponding digit in the constructed number is a 0.)[4]

[4]There is a slight complication: Our rule should *not* simply add one if the relevant digit is an 8 or a 9. The potential problem is that if all the relevant digits to be changed were an 8, then our new, constructed number would be 0.9999 . . . , which mathematicians interpret as equal to 1. The number 1 is outside the range (0, 1), and hence it would be an open question whether the original list contained all of the reals in this range. (By the same token, if the new, constructed number were 0.000 . . . , then Cantor's argument would lose its force.) To avoid this problem, we can amend the rule to *subtract* one from the relevant digit if it is an 8 or a 9 *and* if it is the first digit in the new number; this *ad hoc* amendment will ensure that we never generate 0.999 . . . or 0.000 . . . as the new number. Thanks to Mark Watson for this subtle point.

The point of this construction is that this new number cannot possibly be on our original list. It's not the first number on the list, because it differs from that number in the first digit. Our new number is also not the second number on the list, because these numbers differ in their second digit. In general, the new, constructed number differs from the n^{th} number on the original list in the n^{th} digit. Consequently, no matter how long we search, we know that we can never find it on our original list. And since the constructed number 0.87460 . . . is a real number in the interval (0, 1), we have contradicted our original assumption that we were able to enumerate all such numbers in a list. It therefore must be literally *impossible* to do so, even in principle.

The terminology mathematicians use to distinguish the cardinality (or "size") of these two sets is relevant for the present paper. Both sets of numbers—i.e., the set of integers *and* the set of real numbers between 0 and 1—are infinite; for any finite number one wishes to pick, there are *more* than this many elements in either set. However, as we have just seen, there is a definite sense in which there are "more" elements in the latter set: If we try to put the integers in a one-to-one correspondence with the reals in (0, 1), we will fail, because we can always discover an element of the latter set that does not match up with any integer. For this reason, mathematicians classify the set of integers as "countably" infinite, while they classify the set of real numbers as *uncountably* infinite.[5] The idea is that, given enough time, one could "count up" all of the integers; a person could start writing down 1, 2, 3, 4, . . , and would eventually hit every one. Another way to put this is that "in principle" a person could enumerate or list all of the integers. In contrast, *even given an infinite amount of time*, a person would be unable *even in principle* to enumerate or list all of the real numbers in the range (0, 1). As I now hope to demonstrate, this apparently abstract distinction is quite relevant to the socialist calculation debate.

The Mathematical Solution

Recall that the standard verdict on the socialist calculation debate is that, so long as computational power is not the issue, "in principle" the planners could mimic any market outcome. Yes, as Hayek (and Lionel Robbins) argued, there would certainly need to be millions or billions of equations if the mathematical solution were to be applied to a real world market, rather than a hypothetical model that contains only a few consumer goods. But so what? "In principle" we can characterize such a system of equations and their solution.

The Austrian might object at this point, and wonder how the socialist planners could possibly publish a listing of the billions of prices necessary in this framework. But is this not merely a "practical" objection? The socialist

[5]For a textbook treatment of these issues, see Thomson et al. (2001, pp. 29–31).

theorist could still sit comfortably on his belief that in principle the scheme would work.

What I would now like to argue is that, if the socialist planners really are to mimic the market outcome, they would need to publish a list containing, not merely a huge number of prices, and not merely an infinite number of prices, but rather a list containing an *uncountably infinite* number of prices. But as we have seen above, it is literally impossible, even in principle, for socialist planners to publish such a list. That is, even if we granted them a sheet of paper infinitely long and gave them an infinite amount of time, they still could not, even in theory, write down the entire set of "accounting prices" at which their managers would be required to exchange factors of production. Therefore the purported mathematical solution to Mises's challenge is truly impossible to implement, in every sense of the word.

To understand why the planners would need so many prices, consider the problem of innovation. Back when Hayek and Lange were arguing, there were no market prices for, say, laptop computers. Thus, even if socialism could mimic the market for a few years, it would eventually fall short because it would lack the introduction of new goods that is so typical in market economies.

But wait! The true believer in the theoretical purity and elegance of the mathematical solution has a response. In principle, the planners *could* have included the price of laptop computers back in 1936. Of course, given the state of technology, the supply curve for such a product would have been the y-axis; i.e. no matter how high the "price" (as announced by the Central Planning Board), the producers would have offered 0 units of laptops in 1936. Thus, in order to achieve equilibrium in this market, the announced price would have to be high enough to so that quantity demanded (by socialist consumers) of laptops would also be 0. As Lange argued, presumably this would eventually occur after the initially announced prices (resulting in shortages in the laptop market) guided the planners to continually raise the price until finally excess demand in the laptop computer market were zero.

This technique could be used for *all possible* future goods and services[6]—again, "in principle." For example, the clever entrepreneurs in a capitalist system will no doubt one day provide vacations to Mars. In other words, in the year 2100, there will no doubt be (assuming capitalism is not crushed before then) market prices for shuttle trips to Mars. Consequently, *at that time*, a rival socialist system would need to incorporate the prices associated with such industries into its official list (distributed to managers). Therefore, if a

[6]Strictly speaking, the prices published by the Central Planning Board might not include those for consumer goods, as writers such as Dickinson allowed that these prices would be determined by supply-and-demand. However, there would certainly be at least one higher-good price associated with each consumer good industry. For example, even if the price of laptops were not required in the list of official prices, certainly the (accounting) price of memory boards (used in the laptops) would be required.

socialist system is to be implemented *today*, it needs to have all such prices included in its periodic listing. (Of course, the equilibrium quantities produced and consumed in this industry will also be zero for the foreseeable future.)

The reader probably sees where this is going: Once we realize that all *conceivable* goods and services that might be offered, must have corresponding prices included in the planners' official lists, we understand that such lists would necessarily contain an uncountably infinite number of items. This is inescapable, since after all, eccentric mathematicians in the year 2200 might be willing to pay 1 gram of gold in order to have, say, the number pi written on the night sky (to an arbitrarily long number of digits). For a different example, consider that in order to be sure that socialism wouldn't cheat fiction lovers, it would be necessary to have prices for every single book that could possibly be written in the future. Clearly, Hayek and Robbins grossly underestimated how many equations would actually be required to implement the mathematical solution.

Before closing, let me deal with the obvious socialist retort to the above arguments. Surely, he or she would claim, a socialist planner in the year 2006 would not need to bother with prices associated with trips to Pluto or spare parts for android laser beam eyes. The planner could use his or her common sense and only include prices for goods and services that might realistically be offered in the near future.

Yet such a move would spell the utter defeat of the mathematical solution. Once the socialist relies on the "common sense" of the planners to determine *beforehand* precisely which goods and services are economically relevant, then the socialist has assumed away the very problem at issue. The alleged virtue of Lange's solution was that the planners would not need to "guide" the system in any way, that automatic tinkering of the individual prices in response to shortages or surpluses would, through a blind trial and error process, eventually achieve general equilibrium. To now insist that the planners use their intuition before setting up the system of equations is to concede the entire case.

The last objection I shall consider was raised by both referees, and runs as follows: Why is it necessary for the socialist planners to list all possible goods? Why is it not enough that for any good that is proposed to be produced, the planners can come up with an equation for that good? The socialist can say to the critic, "I can fulfill any task that you can specify. I cannot list an uncountably infinite set of equations, but you can't list an uncountably infinite set of goods for me to produce."[7]

There are two responses to this objection. First, although my exposition above tended to focus on final consumer goods, we should keep in mind that

---

[7]This paragraph is taken almost verbatim from a referee's report on the original version of this article.

perhaps the vast majority of goods produced in a modern economy are actually intermediate products that one firm sells to another firm. The introduction of new goods does not refer exclusively to air conditioned cars, laptop computers, and never-before-written books of fiction, but also includes an industrial sized fluorescent light bulb that uses less electricity, or a new type of insulation that allows a factory to better retain its heat in the winter. Longtime shoppers at Wal-Mart may remember the introduction (a few years ago) of turnstiles for the plastic bags at checkout; the "consumers" of these goods were not the customers of Wal-Mart, but the managers (and ultimately the owners) of Wal-Mart. Thus in a socialist system, it would be up to the managers of various industries to dream up many of the new products and ask the planners to include them in the next batch of official prices.

Second, even if we restrict our attention to final consumer goods, it is not always the case that consumers conceive of a new product and then clamor for entrepreneurs to produce it. As critics of the market economy such as Galbraith point out, advertisers often create a demand for a new product. Depending on one's perspective, this point is quite obvious. For example, when the Wachowski brothers wrote *The Matrix*, were they responding to "given" consumer demand for a movie involving widespread deception by a computer and incredible martial arts footage? Or is it more accurate to say that the Wachowski brothers *invented The Matrix* on their own, and then after the innovation consumers realized how much they loved it?

As the above two responses demonstrate, it is often the case that new products are introduced *not* in response to consumer requests (or more generally, preexisting desires), but are indeed first conceived of by the producers themselves. What reason do we have to suppose, then, that socialist producers would be as inventive as their capitalist counterparts? As Israel Kirzner has argued (e.g., Kirzner 2000, pp. 3–40), one of the crucial advantages of a market economy vis-à-vis a socialist community is that the former can more effectively exploit the "alertness" of entrepreneurs. It is not merely that socialist managers may fail to act upon the socialist analog of pure profit opportunities because of poor incentives, but rather that the managers may *fail to discover the opportunities* in the different institutional framework. To avoid this problem, it is crucial to the success of the socialist project that all of the potential goods be included on the list in the first place, but it is precisely this requirement that is impossible to fulfill.

The socialist theorist cannot have it both ways. Lange's solution to Kirznerian (following Hayek) objections about innovation and risk-taking is to remove the matter completely from the discretion of the planners, and render the entire process one of trial and error. Yet, as I have argued above, this would only work if the prices of all potential goods are included. The socialist cannot then try to overcome *this* objection by invoking the Central Planning Board's understanding of which goods should be introduced and which will be a waste of resources.

Conclusion

The standard view of the socialist calculation debate is that Mises and Hayek at best demonstrated the *practical* impossibility of socialist economy, but that the mathematical solution of economists such as Dickinson showed that "in principle" planners could achieve a rational use of resources without private ownership of the means of production. In the present paper I hoped to show that this view is incorrect, because (if seriously implemented) a socialist planning board would need to publish a list containing an uncountably infinite number of prices. As Cantor's diagonal argument from set theory shows, it is demonstrably impossible to construct such a list. Therefore, socialist economy is truly impossible, in every sense of the word.

References

Dickinson, H.D. 1933. "Price Formation in a Socialist Commonwealth." *The Economic Journal* 43, no. 170 (June): 237–50.

Hayek, F.A. [1935] 1990. "The Present State of the Debate." In *Collectivist Economic Planning*. F.A. Hayek, ed. Clifton, N.J.: Augustus M. Kelley.

Kirzner, Israel. 2000. *The Driving Force of the Market*. New York: Routledge.

Lange, Oskar. 1936. "On the Economic Theory of Socialism: Part One." *Review of Economic Studies* 4, no. 1 (October): 53–71.

Lavoie, Don. 1981. "A Critique of the Standard Account of the Socialist Calculation Debate." *Journal of Libertarian Studies* 5 (1): 41–87.

Mises, Ludwig von. [1935] 1990. "Economic Calculation in the Socialist Commonwealth." In *Collectivist Economic Planning*. F.A. Hayek, ed. Clifton, N.J.: Augustus M. Kelley.

Thomson, Brian S., Judith B. Bruckner, and Andrew M. Bruckner. 2001. *Elementary Real Analysis*. Upper Saddle River, N.J.: Prentice-Hall.