THE MODERN THEORY OF CONSUMER BEHAVIOR: ORDINAL OR CARDINAL?

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The modern approach to consumer behavior, indifference curve analysis (ICA), is based, in theory, upon "revealed preference." In actual practice, each (different) indifference curve is the locus of points generated by setting the differential of a utility function equal to a (different) constant. Neoclassical economists maintain that indifference curves, utility and the utility functions from which they are derived, are ordinal, not cardinal, in nature. They also maintain that utility cannot be measured cardinally. That is, an individual can only prefer A to B or be indifferent between them;[1] he cannot measure how much he prefers A over B. They also maintain that interpersonal utility comparisons are logically invalid.[2] Indeed, the history of economic thought bears eloquent witness to the fact that the concept of ordinal utility triumphed over cardinal utility.

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The author wishes to thank my colleague Walter Block and two anonymous referees of this journal for careful, meticulous and incisive comments on an earlier version of this article. Thanks to their efforts, this article has been deepened, broadened, and reorganized. Although the paper has been vastly improved through their exertions, the author is still responsible for all remaining errors and infelicities.

[1]An important dispute concerns indifference. Austrians (Rothbard 1997, pp. 225–27; and, Block 1980, 1999) hold that an individual cannot, in an economically meaningful way, be indifferent between two bundles of goods, whereas non-Austrians (Caplan 1999; Nozick 1977) maintain they can.

[2]That, as a matter of economics, such comparisons are nonscientific does not obviate the necessity of courts to make them every day. Moreover, in attempting to make such comparisons as scientific as possible for this purpose, courts, *inter alia*, allow "experts'," including economists', testimony into the record upon which triers-of-fact may base their decisions on the quantum of damages. Of course, because the whole process is so subjective it is not uncommon for appellate courts to force the quantum to be revised. For more on this see, Block (1980, 1999).

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Austrian economists also maintain that utility is ordinal. However, they challenge the use of mathematical utility functions by neoclassical economists on the grounds that such functions yield cardinal utilities, "measured," usually, in utils.[3] Neoclassicals respond by asserting that, in dealing with bundles of goods: (1) a function that ranks bundles in accordance with an individual's necessarily ordinal preference ranking is an ordinal function, and the ranking it generates is ordinal; (2) because the ranking of bundles generated by a specific utility function (F) remains the same after any positive monotonic transformation into another function (G), G, also, is a utility function; and, (3) in that case, it does not make any difference whether an individual's preferences are represented by F, by G, or by any other function that is a positive monotonic transformation of F (or of G for that matter).[4]

However, maintaining that "properly constructed" utility functions are valid representations of, admittedly, necessarily ordinal utility is very misleading, and confuses and obfuscates the issue, because as these functions are to be the objects of calculus operations they necessarily require cardinal numbers.[5] Thus, the use of such functions necessarily implies that we are simultaneously in the realms of cardinal *and* ordinal utility,[6] which is problematical, at best. The problem originates in their interpretation of utility functions as ordinal and not cardinal. It is compounded by two other factors: the assumption that utility functions are differentiable; and the failure to use, correctly and consistently, dimensions in their work with utility functions. It is my contention that their analysis is incorrect, both as a matter of praxeology and as a matter of mathematics.

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[3]On the confusing use of "utils" and other cardinal units of measure of utility or happiness by economists who acknowledge that utility or happiness is ordinal, see the appendix, below.

[4]On the issues in this paragraph, see Rothbard (1993, pp. 67–108, 1997, pp. 211–54); High and Bloch (1989, pp. 351–65); Gordon (1992, pp. 99–112); McCulloch (1977); and White (1995).

[5]Austrians do not object to the idea of utility functions, *per se*; i.e., they have no objection to truly ordinal utility functions. Rather, they find objectionable the *de facto*-cardinal neoclassical utility functions, and the uses made thereof.

[6]An anonymous referee comments that,

> It was the use of indifference curves by the victorious neoclassicists that permitted them to have ordinal utility and mathematical functions too. Indifference curves, invented by Edgeworth in the 1880s, made no advance among economists until it was noticed that they made it appear that one could advocate ordinal utility while doing mathematics.

I regard this statement as complementary to my own view, and thank him for it.

The purpose of this article is to demonstrate that neoclassical utility functions[7] are an invalid means of analyzing consumer behavior for three reasons: first, and most important, because such functions, and their attendant rankings, are cardinal, not ordinal in nature; second, because, with respect to the set of bundles relevant to actual human beings, such functions are not continuous and therefore, not differentiable; and, third, such functions do not correctly, consistently, and properly include dimensions/units.

In the next section, I identify the two central elements of the neoclassical position on utility functions, using the views of Samuelson (1965) on this matter; in the sections "Utility Functions Constructed" and "Differentiability and Continuity," I consider each of these elements in turn; in the section "Dimensions," I analyze this issue from the very neglected aspect of dimensions; the final section contains my conclusions.

Central Elements of Neoclassical Utility Theory

Consider the following from Nobel Laureate Paul A. Samuelson (1965, pp. 93–95).

> By the end of the nineteenth century many writers, notably Pareto, had come to the realization that it was an unnecessary and unwarranted assumption that there even exist utility as a *cardinal* magnitude. Since only more or less comparisons are needed for consumer's behavior and not comparisons of how much more or less, it is only necessary that there exist an *ordinal* preference field. For any two combinations of goods $(x_1^0,\ldots,x_n^0)$ and $(x_1^1,\ldots,x_n^1)$, or for brevity (X^0) and (X^1), it is necessary only that the consumer be able to place them in one of the following mutually exclusive categories.
>
> a. (X^0) preferred to (X^1)
>
> b. (X^1) preferred to (X^0)
>
> c. (X^0) and (X^1) equally preferred or indifferent.

[7]An anonymous referee argues that

> The cardinalists . . . claimed that ordinal ranking of bundles could result in either preference or indifference. (The latter) permitted mathematical functions to be used since indifference means equal utility and thus, implies cardinal numbers. Rothbard's (1997, pp. 225–27) response to this is to claim that indifference cannot be the basis for action, only preference can, and therefore, that the assumption of indifference is a psychological, not a praxeological, one.

I do not disagree with this. Very much to the contrary. However, it does not seem likely that the neoclassicals will ever abandon indifference. Therefore, this article is an

For convenience, we may attach a number to each combination; this is assumed to be a continuous differentiable function. This function, or rule of numbering, may be written:

$$\varphi = \varphi\,(X) = \varphi(x_1, \ldots, x_n).$$

It is so constructed that the following three conditions correspond to the above three respectively:

$$a' \; \varphi\,(X^1) < \varphi(X^0)$$

$$b' \; \varphi\,(X^0) < \varphi(X^1)$$

$$c' \; \varphi\,(X^0) = \varphi(X^1)$$

φ may be designated as a utility index[8] [footnote added]. The one-parameter family of loci defined by

$$\varphi(x_1, \ldots, x_n) = C,$$

where C is regarded as a parameter, are designated as indifference loci.

It is clear that any function

$$U = F(\varphi),\; F'(\varphi) > 0$$

defined by any monotonic transformation of φ, is also a utility index. For

$$\varphi\,(X^1) \gtreqless \varphi(X^0) \text{ implies } U(X^1) \gtreqless U(X^0), \text{ respectively.}$$

The converse also holds. Thus, from any one utility index all others can be derived by a suitable functional transformation.

To summarize, our ordinal preference field may be written

$$U = F(\varphi\,(X^1) \gtreqless \varphi(x_1, \ldots, x_n),\; F'(\varphi) > 0,$$

where φ is any one cardinal index of utility.

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attempt to confute utility functions, not by rejecting indifference, but, rather, on other grounds.

[8]A more modern, but essentially no different, definition is: u: X $\to$R is a utility function representing preference relation $\succsim$ if, for all x, y $\in$ X, x $\succsim$ y $\Leftrightarrow$ u(x) $\geq$ u(y), where R is the set of real numbers, X is a set of bundles of goods, and x $\succsim$ y means that the individual values x at least as much as y. See Mas-Colell, Whinston, and Green (1995, p. 9).

> It is clear that the choice of any one numbering system or utility index is arbitrary. The indifference loci are left unchanged by any alteration of the tags attached to each, providing ordinal relationships are maitained.

Let us be quite clear. Neoclassical economists wish to apply calculus to utility functions; i.e., they wish to take their derivatives. In fact, that is the sole purpose for restating preference rankings in the mathematical language of functions. However, for calculus to be applied to these functions it is necessary that the numbers assigned be cardinal.

In order to achieve their purpose, two basic elements, one developed and one assumed, are established, each of which is essential to the claim that, if properly constructed, utility functions can be ordinal *and* grist for the mill of calculus. They are:

(1) that a (utility) function can be constructed that assigns *cardinal* numbers to bundles of goods such that the ranking of the bundles in accordance with these numbers is identical to an individual's preference ranking of the same bundles, and, moreover, such a function and its attendant ranking of bundles are *ordinal* in nature; and,

(2) that such functions are differentiable.

But there are grave problems with this analysis. I examine each of these elements of the neoclassical approach in turn.

Utility Functions Constructed

That a utility function can be constructed that would order bundles of goods consistently with an individual's preference ranking of those bundles is undisputed. The real issues are other.

(1) What is the relationship between such utility functions and actual preferences?
(2) Are such functions and the orderings arising therefrom ordinal or cardinal in nature and how do ordinal preference rankings arise from cardinal preference rankings?
(3) What is the relationship between the nature and the purpose of such functions?
(4) Do results derived from mathematical operations on such functions apply, as well, to the individual's preference rankings that these functions represent?

Ontologically, #1 is probably the most important issue, with #4 second. However, mathematically, and for the purposes of the present article, #2 is the most important, with #3 of lesser importance.

The Relationship between such Utility Functions and Actual Preferences

First, neoclassical economists (Frank 1991, p. 90) maintain that utility functions "represent" preference relationships. Moreover, they have been able to ascertain certain properties of such functions consequent on specific properties of the underlying preference relationships. For example, if a preference ordering exhibits completeness; i.e., if there is a complete binary ordering of bundles such that the individual either prefers bundle A to bundle B ($A \succ B$), or prefers B to A ($B \succ A$), or is indifferent between A and B ($A = B$), for any pair of bundles, A, B; and, if it also exhibits transitivity; i.e., $A \succ B$ and $B \succ C \Rightarrow A \succ C$, and, $A = B$ and $B = C \Rightarrow A = C$, $\forall$ A, B, C, then it can be proved that a utility function exists that represents that preference ordering and that is continuous. However,

> The only thing [representation theorems] establish is that such a function *exists*, not that there is any equivalence between the preference relation and the utility function "representing" it. In other words, they merely permit one to restate, in mathematical terms the verbal conditions expressing preference. (Mahoney 2001, p. 2)[9]

Moreover, utility functions can not generate any information beyond that revealed by the preferences themselves (Mahoney 2001, p. 5).

Second, because in reality individuals' preferences are prior to utility functions, such functions should be developed from observations of individuals' preferences as revealed by their actions.[10] However, in practice, such functions are not so developed. In fact, neoclassical utility functions are constructed for the sole purpose of utilization in mathematical operations, and with a keen eye for properties, that though economically irrelevant, are highly pertinent for mathematical tractability, and with a blind eye to some unavoidable implications. For example, the most commonly used utility function, the Cobb-Douglas (CD), was lifted lock, stock, and barrel from production theory,[11] as were other common utility functions; e.g., the constant elasticity of

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[9]See Mahoney (2001) for an excellent analysis of the issue of "representation theorems."

[10]Although, as neoclassicals maintain, preferences are generally stable in some macro sense, that does not mean they are unchanging in a micro sense. This fact alone challenges the idea that meaningful (necessarily-individual) utility functions can be constructed or developed. Nevertheless, for the purposes of this paper, I ignore this criticism of such functions. Certainly, it cannot be denied that an individual's action demonstrates only preference in that particular context (particularly including that specific time and place); it would be illicit to generalize from any one such choice to any others. I thank an anonymous referee for stressing this point, and have adopted, in part, the language in which he made it.

[11]The CD is also the most commonly used production function. Given that utility is subjective and ordinal, and physical production is objective and cardinal, it is perhaps the most remarkable coincidence in all of economics that one specific function, the CD, is, if we are to judge by the work of neoclassicists, best suited to model both consumer choice and production.

substitution (CES) function. It may well be that a CD is a valid model of some production processes.[12] Indubitably, however, it is an absurdity in the realm of utility theory. Let a generalized CD utility function be given by: $U_j = A\prod(x_{ij})^{\alpha i}$, where i (=1. . .n) indexes the n goods, and j (= 1...m) indexes the m bundles of goods. Then $U_j = U_k$ if any $x_{ij} = x_{ik} = 0$, and $U_h > 0$ if $x_{ih} > 0\ \forall$ i. One property of such a utility function is that if the quantity of at least one good in a particular bundle of goods is zero the entire bundle provides absolutely no utility whatsoever. Then, as between two bundles of goods (A and B), identical in all respects save that A contains one pencil and one house and B contains no pencils but one million houses, the bundle (A) with one house and one pencil is preferred to the bundle (B) with one million houses and no pencils. Moreover, if A contains one pencil and no houses and B contains no pencils and one house, neither would be preferred to the other, nor would either provide any want satisfaction. Therefore, neoclassical utility theory, at least insofar as its most popular manifestation is concerned, must rule out the consideration of any bundles of goods in which the quantity of at least one good is zero.[13]

Moreover, neoclassical economics makes frequent use of the representative agent concept. That is, an imaginary individual is assumed to have a particular utility function, which happens to have the properties necessary to enable the application of calculus. This individual's preferences, as reflected in his utility function, are assumed to be representative of (read identical to) every other person's preferences. The purpose of this is to avoid the virtually impossible task involved in trying to aggregate individual preferences (in the form of utility functions) in some way that would result in mathematical tractability. Again, neoclassical economics starts with utility functions, rather than with actual preferences. A deep yawning gap remains between them, and no need to relate the one to the other is even recognized.

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[12]But even this is questionable for most goods, for surely, with rare exceptions, some output, however small, can be produced with a zero level of some one or more resources that could be used to produce the good; e.g., skilled for unskilled labor or vice versa, and aluminum for steel or plastic, etc.

[13]Additionally, the degree of homogeneity of CD utility functions must be between zero and one. If it is negative, optimization yields a minimum, not a maximum. If it is zero, a scale increase of the arguments does not give rise to any increase in utility. If it is one, a scale increase of the arguments gives rise to the same scale increase in utility; and if it is greater than one, a scale increase of the arguments gives rise to a greater scale increase in utility. None of these results accords with human action. However, if the degree of homogeneity of U is between zero and one, then a scale increase of the arguments gives rise to a less than scale increase in utility. This is another restriction that must be placed on utility functions. That is, the class of acceptable monotonic transformations of CD functions is limited. For example, if the CD utility function F, is homogeneous of degree γ, then any function G = G(F), where $G = F^{\delta}$ and $\partial G/\partial F > 0$, will not do if $\delta \cdot \gamma \geq 1$.

Utility Functions and Rankings: Ordinal or Cardinal?

Neoclassical utility functions and the ranking of bundles they generate are asserted to be ordinal, not cardinal. However, although they can and do order such baskets of goods, they do not use ordinal numbers to do so, rather, they generate cardinal numbers that are then used to rank bundles. They map from the set of bundles to the set of real numbers.[14] But real numbers are cardinal not ordinal. Such functions assign a *cardinal* number to each bundle. That is, neoclassical utility functions generate cardinal numbers and therefore, a cardinal ordering from which an ordinal ranking may be developed. **Moreover, it is not an ordinal ranking that gives rise to the cardinal numbers. Indeed, the ordinal ranking *arises from* the assigned cardinal number; i.e., it is a by-product of the very cardinal number assigned to it.** Therefore, in neoclassical economics the cardinal numbers and rankings are logically prior to ordinal rankings. However, in reality, it is the ordinal ranking which is prior, and which, being prior, makes irrelevant the cardinal numbers assigned to the bundles.

For neoclassical economists the existence of these "ordinal" utility functions makes possible a virtual utopia—they can have their cake and eat it too. They get the truth of subjective valuation in ordinal utility rankings, and, also, the supposed objectivity of cardinal number rankings, which allows them to use calculus. But there can be *only* be an ordinal ranking of utilities. That is, the *only* thing that may be said in comparing two bundles is which is preferred to the other or whether neither is preferred to the other.[15]

That is the crux of the matter. Neoclassical economists act as if they can have both cardinal and ordinal rankings at the same time and in the same respect. That is, they say that because they can order bundles on the basis of the cardinal number assigned to it by a specific function, the order so generated is a rank-order, and that the utilities being so considered are ordinal. This is incorrect. Were it truly an ordinal ranking, it would not be *absolutely necessary* to use the cardinal numbers from which the ranking was generated in their mathematical calculations. Rather, the ordinal numbers corresponding to the ranking generated from the cardinal numbers could themselves be used

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[14]It is important to remember that "function" is a technical mathematical term with a very precise meaning. See, e.g., Thomas (1968, p. 14): "We also speak of a function f from X to Y as a *mapping* that assigns to any element x in its domain a *unique* element y in its range such that the pair (x, y) belongs to f." Note that there is nothing that requires the elements of the range to be cardinal numbers—in fact, they may be ordinal numbers. Therefore, if utility functions were truly ordinal in any meaningful sense of the term, they would map from the set of bundles relevant for an individual to a set of ordinal numbers. However, in practice this is never done, and for good (neoclassical) reason: the calculus operators cannot be applied to ordinal numbers.

[15]Praxeologists do not accept the latter option as valid. Preferences are revealed in, and only in, action. For the praxeological critique of indifference, see Rothbard (1997, pp. 211–54; and 1993, pp. 265–67); Block (1980 and 1999).

in such calculations. Therefore, the order generated by neoclassical utility functions is not a rank ordering in any meaningful sense of the word.

Furthermore, neoclassical economists properly maintain that the ranking of a set of bundles generated by one of their utility functions is invariant under a monotonic transformation thereof.[16] That is, although one bundle of goods from a set of bundles might be assigned the number 100 by one function, F, and the number 10,000 by another function, F^2 (a monotonic transformation of F), in either case the bundle would have the same ranking within the set of bundles. Therefore, in order to know the rank of a basket of goods, all one needs to know is its assigned number, and the numbers assigned to other bundles. Note well that all that is necessary to know in order to assign the appropriate number to each and every specific bundle is its own specific contents. No knowledge whatsoever of the contents of any other bundle or bundles is necessary.[17] Therefore, baskets of goods may be ranked without any comparison with the contents of other bundles. Most assuredly, such a ranking does not qualify as ordinal in nature. Consider the following example. Let:

1. x_i (i = 1...n) be the set of n goods that are relevant for an individual;
2. $X = X_j(x_i)$ (j = 1...m) (i = 1...n) be the set of m different bundles of the n goods that are relevant for an individual;
3. $X_j \succsim X_k$ be a preference relation that means bundle X_j is preferred at least as much as bundle X_k;
4. U be a neoclassical utility function such that for any bundle of goods, $X_i(x_i, y_i)$, $U(X_i) = 10x_i^{0.25}y_i^{0.25}$;
5. $U(X_1) = 80.07^+$;
6. $U(X_2) = 79.93^-$;
7. rank of X_1 = k, where k is an ordinal number (e.g., 3rd or 15th); and,
8. rank of X_2 = k+1.

Now consider an additional bundle, $X_{M+1}(16, 256) \Rightarrow U(X_{M+1}) = 80$. All that it is necessary to know about bundles X_1 and X_2 in order to rank correctly bundle X_{M+1} is that $U(X_1) = 80.07^+$, $U(X_2) = 79.93^-$, the rank of X_1 = k, and the rank of X_2 = k +1. That is, one need know nothing about the elements of bundles X_1 and X_2 (i.e., x_1, y_1, and x_2, y_2)[18] in order to rank X_{M+1}.

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[16]The use of monotonic transformations to buttress the claim that neoclassical utility functions and their attendant rankings are ordinal is considered, *infra*.

[17]It is my view that, in the words of a referee, "value cannot be attributed to a bundle without comparing to other bundles." The process of evaluation is essentially a comparative one; it is picking one thing and setting aside another. If, *arguendo*, there were only one thing to be chosen, the implication is that valuation could not occur. This, of course, cannot take place, because there is always the option between choosing and forgoing.

[18]In fact, $X_1 = X_1(16, 257)$ and $X_2 = X_2(16, 255)$.

Compare that to a true ordinal utility function on the same two goods.[19] Let: V = V(X) be a true ordinal utility function, such that:

$V(X_j) = 1^{st}$ if $X_j \succsim X_k\ \forall\ k \neq j;\ k \in X$,
$V(X_h) = 2^{nd}$ if $X_h \succsim X_k\ \forall\ k \neq j, h;\ k \in X$
$V(X_g) = 3^{rd}$ if $X_g \succsim X_k\ \forall\ k \neq j, h, g;\ k \in X$

$V(X_m) = m^{th}$ if $X_m \succsim X_k\ \forall\ k \neq j, h, g, \ldots m-1;\ k \in X$

Again consider the same additional bundle, $X_{M+1}(16, 256)$. In this case there is no way to know V(M+1) and, therefore, no way to rank X_{M+1}, save by comparing the *elements* of bundle X_{M+1} with the ranking **and** *elements* of other bundles. That is, even in a "best case" scenario, it is impossible to rank X_{M+1} absent knowledge of the elements of two other bundles.[20] It simply will not suffice to know that the ranks of bundles X_1 and X_2 are k and k+1, respectively.

Put another way, let there be two (2) bundles, A and B, to which a neoclassical utility function assigns the cardinal numbers 20 and 30, and ordinal numbers (*necessarily* based thereon) 2nd and 1st, respectively. If a third bundle, C, is now to be ranked, all that is necessary to know are the elements of bundle C and the concomitant cardinal number assigned to it by the utility function. Thus, if the utility function, operating on the elements of C, yields the cardinal number 25 for C, then the ordinal ranking becomes: B is 1st, C is 2nd, and A is 3rd. If, however, the utility function were a "true" ordinal utility function, in order to rank C, we would need to know more than that the ordinal numbers assigned to A and B are 2nd and 1st, respectively, and the elements of C. We would need to know the elements A and of B, as well.

Consider yet another example. Let there be four (4) bundles of goods, A, B, C, and D, with quantities of x and y in each as depicted[21] in Table 1. Because more is better,[22] we know B is preferred to A, C to B, C to A, D to C,

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[19]This particular ordinal utility function allows for indifference; i.e., there may be multiple firsts, seconds, etc.

[20]In a "best case" scenario, by sheer coincidence, the first two bundles that one would compare the elements of the additional bundle with would be X_1 and X_2 with the result: $X_1 \succsim X_{M+1} \succsim X_2$. Therefore, $V(X_1) > V(X_{M+1}) > V(X_2)$. And, because X_1 and X_2 happen to be ranked sequentially, k and k+1, respectively, the rank order of X_{M+1} is necessarily k+1, with the rank order of X_2 and all lesser ranked bundles adjusted accordingly.

[21]The cardinal utility values in Table 1 were generated using two (2) CD utility functions: $U1 = 2(x^{1/2}y^{1/2})$; and, $U2 = V + 2W(x^{1/2}y^{1/2})$ $(V \geq 0, W \geq 1)$, a specific (positive) monotonic transformation of U1.

[22]Actually, for CD type; i.e., multiplicative, functions, "more is better" is not always true, as was demonstrated, above (although it is true in this particular example). That is, in terms of neoclassical utility functions, more utility is better than less, but that is a trivial statement; if, however, the concern is for quantities of goods, then it is not at all clear that more is better; rather, whether more is better depends crucially on how "more" is

D to B, and D to A. That is, in ordinal terms, we rank D first, C second, B third, and A fourth. And this is true whether we use U1 or U2; i.e., the ranking of each bundle is the same regardless of which function is used to rank them, or if no function is used to rank them.

Table 1

Bundle	x	y	U1	ΔU1	U1 Rank	U2	ΔU2	U2Rank
A	100	100	200	-	4th	V + 200W	-	4th
B	121	100	220	20	3rd	V + 220W	20W	3rd
C	400	400	800	-	2nd	V + 800W	-	2nd
D	420.25[23]	400	820	20	1st	V + 820W	20W	1st

Whether using U1 or U2, a neoclassicist would rank the bundles from most to least preferred using their cardinal numbers: 820, 800, 220, and 200 or V + 820W, V + 800W, V + 220W, and V + 200W, respectively. In either case he would rank them D, C, B, and A. Moreover, the mainstream economist would not even assign their ordinal numbers to them. An Austrian would rank the bundles in the same order. However, in sharp contrast, he would not do so on the same basis—he knows there can only be an ordinal ranking of utilities. That is, the *only* thing that may be said in comparing two bundles is which is preferred to the other or whether neither is preferred to the other.[24] Therefore, in the example above, Austrians maintain that the *only* things that may be said are: B is preferred to A, C to B, C to A, D to C, D to B, and D to A, but he cannot compare the relative rankings in any mathematically meaningful way.

Let us be perfectly clear on this matter: the numbers 200, 220, 800, and 820, and V + 200W, V + 220W, V + 800W, and V + 820W, have no more meaning *in reality* then would any other set of four numbers ordered from smallest to largest. Austrians know this *and* they conform their analysis thereto; i.e.,

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defined. Thus, if there is a zero quantity of any one good in a bundle, the total utility of that bundle is zero, no matter how great are the quantities of the other goods in it; i.e., the marginal utility of all the other goods in that bundle must be zero. All of which is a consequence of neoclassicists using mathematics and forgetting about underlying preferences. Austrians know that for almost any individual, if he has zero pencils and one steak in a basket of goods, that adding another steak would entail positive marginal utility, save, perhaps, for some vegetarians. However, for neoclassicists using CD type functions, their math won't admit of this reality. They would have to add an addendum to their functions that would define the method of calculating utility when there is a zero quantity of one or more goods in a basket.

[23]Referring to fractions, think of x as some good measured in kg.

[24]As mentioned in note 15, Austrians do not accept the latter option as valid. Preferences are revealed in, and only in, action.

they eschew the use of neoclassical utility functions, and the cardinal numbers generated thereby.[25] This the neoclassicist cannot do, for in spite of paying lip service to the ordinal nature of utility, in practice, the (implicit or explicit) cardinal numbers generated by their utility functions[26] are at the core of their analyses.

Consider that in the above example, using U1, the mathematics enable us to say that B is preferred to A by 20 (whatever that means) and D to C by 20, and therefore, B is preferred to A by exactly as much as D is preferred to C. But if utility is truly ordinal then no meaningful statement may be made about the intensity of preferences. It is logically impossible to say that: B is preferred to A by 20; D is preferred to C by 20; and, therefore, that B is preferred to A by exactly as much as D is preferred to C. That is exactly the type of fallacy that results from using cardinal utility functions.

Furthermore, neoclassicists make use of indifference curve analysis and the marginal rate of substitution (MRS) to analyze consumer choice. The MRS in the two-good case is defined as: $dy/dx_{(dU = 0)} = -U_x/U_y$.[27] Therefore, using either U1 or U2 as the utility function, $dy/dx = -y/x$. However, utility maximization requires that the MRS be equal to the negative of the ratio of the price of x to that of y ($-p_x/p_y$). Therefore for either bundle A or C to be optimal the prices of x and y must be the same and, therefore, the MRS = -y/x = -100/100 = -1, whereas for B or D to be optimal the price ratios of x to y must be -100/121 or -400/420.25, respectively and, therefore, the MRS = -y/x = -100/121 or -400/420.25, respectively. But -1, -100/121, and -400/420.25 are cardinal numbers, not ordinal numbers. It is difficult to see how this can be denied. **Consequently, one reason that neoclassicists are in error is precisely because, *de facto*, they switch from ordinal to cardinal utility when their utility functions generate cardinal numbers and they use these cardinal numbers in their calculations.**

Nor can the neoclassicists legitimately deny that the cardinal numbers are exactly that—cardinal numbers subject to mathematical operations. In fact, they use the cardinality of the numbers generated by utility functions in their calculations of the MRS. That is, for U = U (x, y), the MRS, dy/dx, is the ratio of the cardinal numbers, dy and dx, and, therefore is itself a cardinal number. (It most certainly is not a ratio of ordinal numbers, itself an oxymoron.) However, having used the cardinality of dy and dx for their own purposes, they wish to deny that this cardinality may be used for other purposes; e.g.,

[25]They also eschew the use of true ordinal utility functions on the grounds that they add no information, but merely restate preferences in mathematical symbols.

[26]The marginal rate of substitution (MRS), which is at the center of indifference curve analysis, is, in the simplest case of two goods, the ratio of two cardinal numbers.

[27]Yet another way to perceive that neoclassical utility functions are cardinal in nature is as follows. Because $dU = 0 \Leftrightarrow U = \text{constant}$, U must be cardinal, as only cardinal numbers can be constants; i.e., ordinal numbers cannot be constants; e.g., 10 is a constant whereas 10th is not.

to compute differences. In effect they claim that the nature of the numbers generated by their utility functions is contingent—it oscillates between cardinal and ordinal as the operation is division or subtraction, respectively, and therefore these numbers may be used to calculate ratios, but may not be used to calculate differences. For example, in effect for them, because x and y are cardinal numbers, dy/dx is a legitimate mathematical operation that generates a cardinal number, but, because (they maintain) $U(x_1, y_1)$ and $U(x_0, y_0)$ are ordinal numbers, $U(x_1, y_1)$-$U(x_0, y_0)$ is not a legitimate mathematical operation and does not generate a legitimate cardinal number.

Moreover, the most common "proof" that neoclassical utility functions are ordinal in nature is based on positive monotonic transformations. Neoclassicists maintain that because the ranking of bundles generated by a neoclassical utility function is maintained under any positive monotonic transformation thereof, totally regardless of the cardinal numbers assigned the bundles by the different functions, that the rankings are ordinal. Consider, for example, two such functions F and G, with G a positive monotonic transformation of F, and two bundles of goods A and B. The relationship between the numbers assigned by F to A and B is arbitrary with respect to the relationship between the numbers assigned to A and B by G, save that as A is greater than, equal to, or less than, B for F, so also will it be for G. That F and G assign different cardinal numbers to A and B is irrelevant for their ranking. Moreover, the MRS is invariant between F and G.[28] That is, regardless of whether A and B are ranked by F or by G, the ranking will necessarily be the same, as will the MRS. Because the relationships between the cardinal numbers assigned by different functions (in this example, F and G) are arbitrary and yet the ranking is maintained, and also because the (cardinal) MRSs[29] are identical, neoclassicists claim that the ranking is ordinal, and, *de facto*, that such utility functions are ordinal in nature and are dealing with ordinal utility. That is, they claim that the use of neoclassical utility functions is acceptable in a world where cardinal utility is recognized as methodologically invalid.

However, that such functions yield cardinal, not ordinal, rankings can be seen from the following thought experiment. Let there be 10 bundles of goods, X_i (i = 1,..., 10), such that the neoclassical utility function happens to assign the cardinal numbers $10 \cdot i$, to these bundles, not necessarily respectively; i.e., bundles X_1 X_2, X_3,..., X_{10} are assigned the cardinal numbers 10, 20, 30,..., 100, though not necessarily respectively. Then an 11th bundle, X_{11}, can be ranked without knowing anything about the contents of any of the others. The cardinal number may be assigned without any knowledge of the contents of the other bundles. If the number assigned to it is, say, 35, then one knows

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[28]The MRS for F and G are $F_i/F_j \ \forall$ i, j and $G_i/G_j \ \forall$ i, j, respectively. But, $G_i/G_j = G_F \cdot F_i/G_F \cdot /F_j = F_i/F_j$, QED.

[29]That the MRSs are cardinal numbers is a fact that is never mentioned by the neoclassicists; the only thing they note is that they are identical, as between different utility functions representing the same set of preferences.

instantly that it ranks immediately below (above) the bundle to which the number 40 (30) has been assigned. The cardinal number can be assigned without comparison to other baskets of goods. For example, given a CD function and a bundle one can immediately know its cardinal number and can immediately tell where it ranks in the order. If its CD# = 35, then it is between 40 and 30, but if 40 is ranked 7th and 30 is ranked 8th, then 35 is now ranked 8th, and the ranking of other baskets of goods is adjusted, *mutatis mutandis*. Therefore, it is unnecessary to compare the elements of X_{11} with those of any other bundles in order to rank the bundles.

Compare that to a true ordinal utility function that assigns the ordinal numbers first through tenth to these same bundles, again, not necessarily respectively. Then one has no idea where the 11th bundle ranks. That is, without specific knowledge of the content of the other bundles, one cannot ordinally rank it, for there is no way to assign an ordinal number to it save by comparison of its contents to that of other bundles. That is, given an ordinal function and a bundle one cannot immediately know its ordinal number or immediately tell where it ranks in the order. X_i must be compared to X_j; if X_i is preferred to X_j it must then be compared to a bundle with a higher ordinal ranking, if lower, then lower.[30] This process (i.e., these comparisons) must be repeated until the exact rank order of X_i is established.

Alternatively, consider a neoclassical utility function, $U = 10\, x^{0.25}y^{0.25}$. If we know that bundle X_1 consists of $x = 16$, $y = 256$, then we know that $U = 80$. However, 80 is a cardinal number that tells us nothing about the rank of X_1. The utility function has not provided any economically meaningful information. But how can an ordinal utility function tell us nothing about rank? In fact, the rank order is precisely the information that would be produced by a true utility function. Rank order implies at least two things to be ranked. When there is only one, then, perforce, there can be no rank, as in the present case.

To conclude this section I note that there is no reason in mathematics, and certainly none in logic or the analysis of human action, that requires utility functions to assign cardinal numbers to bundles of goods. In fact, not only could ordinal numbers be assigned, they *should* be assigned instead of cardinal numbers. That is, because utility is, admittedly,[31] ordinal, if it is desirable to express preference rankings among different bundles of goods as mathematical functions, such functions should assign the appropriate *ordinal* number to each bundle, rather than a cardinal number and then maintaining that the only importance thereof is its use in ranking bundles. Why bother assigning cardinal numbers that then have to be interpreted as ordinal, i.e.,

[30]Only if by sheer coincidence neither is X_i preferred to X_j nor is X_j preferred to X_i does the process of comparison stop and X_i receives the same ordinal number as X_j, with the ordinal numbers/ranking of all lower numbered/ranked bundles being adjusted, accordingly. The usual caveats about indifference for Austrians still apply, however.

[31]By both Austrians and neoclassicals, presumably.

converted to ordinal numbers, when ordinal numbers could be assigned in the first place? Occam's razor would certainly favor first using ordinal numbers in this case. The answer, of course, is that the only (neoclassical) reason for assigning the cardinal number is to prepare the way for the use of calculus.

The Relationship between the Nature and the Purpose of such Functions

Neoclassical economists wish to make economics scientific in the same way as are the other "hard" sciences such as physics. They desire to develop mathematical models of that which is being studied (purposeful human action, in the case of economics) and then use data to test hypotheses arising from the theory. Although neoclassicists in economics also use other tools of mathematics, calculus is the key subdiscipline, because it allows calculation of *maxima* and *minima*. This is essential, because of the importance to economics of such concepts as utility and profit maximization and "cost" minimization. However, the very use of calculus requires cardinal numbers; calculus cannot possibly be used in conjunction with ordinal numbers (Thomas 1953, pp. 11–14). Moreover, in order to obtain more than a few useful generalizations[32] from the use of calculus, specific functional forms, with attendant specific cardinal implications, must be employed. In utility analysis this means a cardinal number must be assigned, at least implicitly, to each bundle.

Consider, then, the sole purpose for which utility functions are constructed: to manipulate them mathematically, and especially to determine the conditions for the maximization of utility subject to a budget constraint. This usually shows up as a trivial calculus problem: maximize $U = U(x_1,\ldots,x_n)$ subject to $B = \Sigma p_i x_i$ ($i = 1\ldots n$), where U is the utility function, x_i ($i = 1\ldots n$) are the n goods upon which utility depends, and p_i is the price of a unit of x_i $\forall$ i, and B is the fixed budget constraint.[33] The solution, using standard notation, is: $U_i/U_j = p_i/p_j\ \forall\ i, j, i \neq j$. Of course, p_i/p_j is the objective exchange rate—the rate at which people can trade x_j and x_i for each other. Regardless of whether one interprets U_i/U_j as the ratio of two marginal utilities or as the MRS[34], it is a subjective exchange rate[35]—the rate at which people desire to trade x_j and x_i for each other. Now, it is obvious that the magnitude of p_i/p_j is a cardinal number; for example, let $p_i = \$10/x_i$ and $p_j = \$5/x_j$, then $p_i/p_j = 2x_j/1x_i$. That

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[32]It should be noted that the calculus is not necessary to these generalizations; they can be reached quite easily through simple deductive logic.

[33]Sometimes a time index is included, but this is irrelevant for the point being made.

[34]In the case of three or more goods, the MRS = dx_i/dx_j, x_k = constant $\forall\ k \neq i, j$.

[35]If one interprets U_i/U_j as a ratio of marginal utilities, presumably the dimensions of U_i and U_j would be utils/x_i and utils/x_j, and consequently the dimensions of U_i/U_j would be x_j/x_i. Alternatively, if one interprets U_i/U_j as the MRS ($-dx_j/dx_i$) presumably the dimensions of dx_i and dx_j are x_i and x_j, respectively, and the dimensions of U_i/U_j would then be x_j/x_i. In neither case is U_i/U_j dimensionless. But if U_i/U_j has dimensions, they are necessarily cardinal, not ordinal.

is, the rate at which x_j can be traded for x_i is $2x_j$ for $1x_i$. Moreover, because $U_i/U_j = p_i/p_j$, and $p_i/p_j = 2x_j/1x_i$, U_i/U_j cannot be equal to 2/1, because $2/1 \neq 2x_j/1x_i$; rather U_i/U_j must be equal to $2x_j/1x_i$. But, even if the laws of mathematics were somehow suspended to allow 2/1 to be equal to $2x_j/1x_i$, 2/1 is still a cardinal number, two; i.e., it is twice one, not merely greater than one, and it is one-half four, not merely less than four. And, 2/1, most assuredly, is not second x_j for first x_i. Put differently, there are no ordinal numbers involved. If, for example, x_i and x_j are bushels of wheat and corn, respectively with $p_i = \$10/x_i$ and $p_j = \$5/x_j$, then the first order conditions for utility maximization, $U_i/U_j = p_i/p_j$, require that $U_i/U_j = 2x_j/1x_i$ (i.e., $2x_j$ trades for $1x_i$); however, this is cardinal/objective not ordinal/subjective. That is, the ratio of the marginal utilities when utility is maximized is 2 bushels of corn to 1 bushel of wheat, **not** plain old two to one or *second* bushel of corn to *first* bushel of wheat. This is hardly ordinal or subjective analysis; in fact utility cannot be both subjective and cardinal.

Nor does it make any difference if the solution is written $U_i/p_i = U_j/p_j \ \forall$ i, j, i ≠ j. If one interprets the $U_i \ \forall$ i as dimensioned marginal utilities, presumably the dimensions of U_i and U_j would be utils, and consequently the dimensions of U_i/p_i and U_j/p_j would be $(\text{utils}/x_i)/(\$/x_j)$ = utils/\$ and $(\text{utils}/x_j)/(\$/x_j)$ = utils/\$, respectively. In this case, utility has the dimension utils, and therefore utility is necessarily cardinal, not ordinal, in nature. Alternatively, if one interprets the $U_i \ \forall$ i as nondimensioned marginal utilities the dimensions of U_i/p_i and U_j/p_j would be $1/(\$/x_j)=x_i/\$$ and $1/(\$/x_j) = x_j/\$$, respectively. But then because $U_i/p_i = U_j/p_j \ \forall$ i, j, i ≠ j, we arrive at the conclusion that, in terms of dimensions, $x_i/\$ = x_j/\$ \Rightarrow x_i = x_j$. Therefore, the dimensions of two different goods x_i and x_j are identical; i.e., the two *different* goods are the *same* good. Moreover, as the condition holds for all goods, *each and every different good must be the same good*.

The two main conclusions derived from the use of utility functions are that utility maximization and, *a fortiori*, equilibrium for the consumer require that: (1) the MRS between any two goods be equal to the negative of the ratio of the prices of the two goods; and (2) the ratio of the marginal utility to the price of any good be equal to the negative of the same ratio for any other good. However, as proven in this section, both of these conclusions are *mathematically invalid*.

Applicability of the Results Derived from Mathematical Operations on Neoclassical Representative Functions to Individual's Actual Preference Rankings

Neoclassical economics has various "representation" theorems supposedly used to prove that the results of mathematical operations applied to utility functions pertain also to the underlying preferences of the relevant individual. However, these theorems do nothing of the kind. They merely prove that such functions can be constructed and, that given certain characteristics of the underlying preference relationship, they have certain (mathematically) desirable properties. Of fundamental importance is the fact that these theorems in no way prove that the utility function is equivalent to the preference ordering it represents. Therefore, it is invalid to assert or imply that (any) results obtained by means of mathematical operations applied to such functions necessarily hold true for the underlying preference relationships. More generally, the best that may be said is that utility functions are merely restatements of preference rankings in the language of mathematics.

Suppose, furthermore, that somehow neoclassical utility functions were equivalent to the underlying preference relationships, such that mathematical results derived from these functions applied to the real relationships. Then, because these functions are inherently cardinal, the inevitable conclusion would be that individuals' preferences themselves are also cardinal. But neoclassical economists, themselves, have long since concluded that individuals' preferences are ordinal, not cardinal.

Additionally, as noted above, in making interpersonal utility comparisons, as they must, courts make use of experts' testimony. And so also do legislatures and governmental agencies in their work, that also, necessarily involves such comparisons. It stands to reason that if neoclassical utility functions actually represented people's preferences, and the mathematical results derived therefrom applied to the real world, such functions would make their way into expert-economists' testimony—testimony that is both impeachable and for which economists can be held liable for perjury. What is certain is that neoclassical economists do not introduce such functions into their testimony. The reasons are obvious.

Differentiability and Continuity

Consider the issue of continuity. Neoclassical economists apply calculus to utility functions in order to analyze human action. This necessitates that these functions be differentiable, and, therefore, continuous.[36] Therefore, they do not develop their utility functions by making empirical studies of individuals' preferences (as revealed by their actions) in order to determine specific functions that would or do represent individuals' preferences.[37] Rather, they

[36]"Differentiability implies continuity" (Thomas, 1968, p. 99). It should be noted that, although much is made of continuity in the neoclassical literature referring to differentiability, continuity, though a necessary condition, is not a sufficient condition for differentiability.

[37]This is not to deny that many of the functional forms used in *econometrics* were chosen after preliminary analysis of data. However, the set of forms from which the chosen forms were/are taken was/is not open ended; rather, it was/is restricted to those that are differentiable and tractable, and also (directly, indirectly, or at least, approximately) estimable. It should be noted that *any* known set of preferences can be exactly represented by an appropriately chosen polynomial function. For example, if A's utility depended

merely choose either a nonspecific, assumed-differentiable function, or certain specific types of differentiable functions.[38] In the former case, the functions are assumed to have certain desirable properties, including continuity; in the latter cases the actual functions chosen exhibit the desirable properties, including differentiability. (It should be noted that one of the desirable properties for specific functions is mathematical tractability.)[39]

However, utility functions are supposed to be representations of real *individuals'* actual preferences, and, undeniably, many of the goods that are arguments of individuals' utility functions are discrete; e.g., automobiles, houses, furniture and appliances, etc. Therefore, of necessity, the utility functions are discontinuous.[40] Thus, the very (discrete) nature of many goods is all by itself sufficient grounds to invalidate the uses which neoclassicists make of utility functions. Neoclassicists evade this aspect of reality[41] through the simplistic

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upon the quantities of n goods, and A ranked z bundles of the n goods, a utility function could, in principle, be found that would exactly represent the ranking of the z bundles. (It is immaterial to statistics, but not to economics/praxeology, whether such bundles purport to be contemporaneous alternatives or, instead to exist at differing points of time.) Put another way, in the n+1 (utility plus the n goods) space a utility hyper-surface could, in principle, be found such that each and every one of the (z) data points corresponding to the z bundles would lie precisely on the utility hyper-surface. Such a function is of no use in econometrics because, having no degrees of freedom, there are no unknown parameters that need be, or can be estimated; i.e., all is known, there is nothing to be estimated, no work for statistics to do. Moreover, in practice, the normal course of events is to take one, or at most a few of the standard functional forms and, using various statistical techniques, attempt to determine which fits the data best, based upon various criteria; i.e., the function(s) is/are picked first and only then are the data fitted thereto. In many or most cases the data are selected and are modified/transformed based on the restrictions imposed by econometric necessity.

[38]For example, CES (including CD) functions.

[39]That is, the functions actually selected are chosen, not because they actually represent real people's preferences, but rather, because they allow calculus to be applied, and that in a mathematically tractable way; i.e., they do not choose to use a specific differentiable function unless it also happens to be mathematically tractable.

[40]It is true that some noncontinuous functions could be treated as continuous provided that, relative to the relevant values of x, the minimum value Δx could take on was sufficiently small so that $\Delta x/x$ was of an insignificant magnitude. For example, for the Cobb-Douglas function $U = c^{\alpha}h^{\beta}$, where c and h are cars and houses, respectively, and therefore, the minimum value that either Δc or Δh may take on is one (1), if the relevant values of c and h are in the neighborhood of 1,000,000, such that $\Delta c/c \approx \Delta h/h \approx 0.000001$, the function could be treated as if it were continuous, and the partial derivatives taken and evaluated. However, in the real world, the relevant values of c and h are much more likely to be integers such as one or two, and certainly, save for a very few people, rarely would exceed five. Certainly for values of $\Delta c/c$ and $\Delta h/h \geq 0.2$ this function cannot be treated as continuous.

[41]What of the argument contra, "What's the harm?" or "No harm, no foul." or "Yeah, but so what?" That is, continuous and differentiable cardinal functions indeed are unrealistic, but they make it easier to understand a few basic points, such as consumer equilibrium. In this view, the assumption of continuous and differentiable cardinal functions is

expedient of assuming that their utility functions are continuous and differentiable.[42]

DIMENSIONS[43]

This brings us to the problem of dimensions.[44] (In the analysis that follows, I attach the relevant units,[45] in brackets, to the variables as appropriate.) Let: U = (x [kg])$^{0.25}$(y [m^2])$^{0.25}$; P_x = \$5/kg; P_y = \$20/m^2; and, B = \$1,000; where: U

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like a heuristic device, or a rule of thumb; e.g., when calculating the momentum of a body, the formula p = mv, where p is momentum, m is mass, and v is velocity, provides an excellent approximation to the correct value, unless the speed of the body is a significant fraction of that of light.

The problems with this response are several, and serious. First, this is not at all how the concept of continuous and differentiable cardinal functions operates in economics. Nowhere is it ever stated that these are merely approximations, and not fully scientifically accurate; rather, they are paraded as not only entirely valid, but as constituting the very litmus test for sophisticated economics. One reason Austrian economics is rejected by mainstream practitioners is due to the supposed lack of mathematical sophistication of the latter. Second, economists have been led by these improper assumptions to fallacious public-policy recommendations, for example, antitrust. It is only a slight exaggeration to say that the entire intellectual case for antimonopoly legislation is based on the foundation of continuous and differentiable cardinal functions. For an explicit analysis along these lines see Rothbard (1993, p. 644). For other studies which make use of this insight see Anderson, et al. (2001); Armentano (1991); Block (1994); DiLorenzo (1997); Boudreaux and DiLorenzo (1992).

[42]For other Austrian critiques of neoclassical mathematical economics see Mises (1977); Leoni and Frola (1977); Rothbard (1993).

[43]On the failure of neoclassical economists to consistently and correctly use dimensions/units in their work, see Barnett (forthcoming).

[44]Schumpeter (1986, p. 1062, n. 1) was aware of the *necessity* of defining units in order to measure things.

> A quantity or magnitude (the Greek μεγυος) is defined as anything that is capable of being greater or smaller than some other thing. This property implies only transitivity, asymmetry, and aliorelativity (the last term meaning that no thing can be greater or smaller than itself). It also covers the relation of equality, which is however symmetrical and reflexive (the latter term meaning the opposite of aliorelative). Now, quantity in this very general sense does not imply measurability, which requires fulfillment of two more conditions: (1) **that it be possible to define a unit;** (2) that it be possible to define addition *operationally*, i.e., so that it can actually be carried out. (emphasis added)

In an otherwise scathing attack on Cassel's theory of interest, Böhm-Bawerk (1959, pp. 196–201) acknowledges that the correct use of units is a necessary, but not sufficient, condition for economic theories. The author thanks Roger Garrison for pointing out this citation to me. Also see Garrison (1988, pp. 49–52) in this regard.

[45]The units used, from the international system of units–Le Systéme Internationale d' Unites, are kilogram (kg) for mass and square meter (m^2) for area, and for money the dollar.

is the utility function; x is food, measured in kg; y is shelter, measured in m^2; P_x and P_y are the prices in \$/kg and \$/m^2 for food and shelter, respectively; and, B is the budget constraint.

The budget constraint equation is: \$5/kg • x[kg] + (\$20/ m^2) • y[m^2] = \$1,000 and its slope is: $dy_{(B)}[m^2]/dx[kg] = -0.25\ m^2/kg$. Of critical importance is that the slope of the budget constraint is in terms of m^2/kg. It is, most emphatically, not a pure magnitude. And this dimensioned term has real economic meaning—in the market one can exchange food for shelter at the rate of four kilograms of food for one square-meter of shelter, and vice versa—which meaning would be totally absent were it not for the units. That is, if the slope was solely -0.25, as opposed to -0.25 m^2/kg, no economic meaning whatsoever would attach to it.

Of course, neoclassical analysis requires for utility maximization that the utility curve be tangent to the budget line; i.e., that the slopes of the utility curve and the budget line be equal.[46] The slope of the utility function is: $dydx(_U) = -y/x$. However the only way that -y/x can be equal to -0.25 m^2/kg is if y has the unit m^2 and x has the unit kg.[47] Moreover, because the slope of the budget constraint *necessarily* has dimensions, if the slope of the utility function does not have the *same* dimensions, or *a fortiori*, if it has no dimensions at all, the two are necessarily incommensurable. That is, there cannot be a solution to the maximization problem. Without dimensions, to claim that there can be a solution to the maximization problem is of the nature of saying that, e.g., 100 (or perhaps 200) can be exchanged for a ton of steel in the market. One hundred WHAT? one wants to shout. But if the utility function does not have dimensions, there cannot be any answer to the question. It is simply 100 (or 200). But there is the rub. For if neoclassical utility functions have dimensions, as they *must*, if their values are to be maximized, which is the primary, if not sole, purpose for their existence, then they are of necessity cardinal. That is, they are cardinal and not ordinal. For the very essence of dimensions/units is cardinality; i.e., the ability to make cardinal comparisons.

Summary and Conclusion

In sum, the cardinal utility numbers generated by neoclassical utility functions provide more information than do their ordinal counterparts. In fact, for any given set of bundles they contain all of the information implicit in ordinal utility numbers for the same set, plus they provide additional information concerning the intensity of the preference for any one bundle relative

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[46]In cases of more than two goods, the requirement is that the utility (hyper-) surface be tangent to the budget (hyper-) plane; i.e., that $U_i/U_j = p_i/p_j\ \forall\ i, j$.

[47]Alternatively, from a purely mathematical perspective, y could have the units m^2/kg and x could be dimensionless, or x could have the units kg/m^2 and y could be dimensionless, or y could have the units 1/kg and x could have the units 1/m^2, but, from an economics perspective, these are absurd.

to any other. It is precisely because utility functions cannot be used to calculate ordinal rankings of bundles without prior calculation of their cardinal utility numbers that the use of utility functions is unacceptable for economic purposes. Moreover, although meaningless with respect to the reality of actual individuals' preferences, this extra information is harmful because it is misleading.

I conclude by reiterating the purpose of this article. I have attempted to demonstrate that neoclassical utility functions are an invalid means of analyzing consumer behavior for three reasons: first, and most important, because such functions, and their attendant rankings, are cardinal, not ordinal in nature; second, because, with respect to the set of bundles relevant to actual human beings, such functions are not continuous and, therefore, not differentiable; and, third, because such functions do not correctly, consistently, and properly include dimensions/units.

Let me put this in another way. I will accept the validity of utility functions as soon as its proponents can show me how to perform basic mathematical or arithmetic operations on such ordinal numbers as 1^{st}, 3^{rd}, 6^{th}, and 17^{th}.

Addendum

Consider the following thought experiment. There are four sheets of paper, numbered one through four. There is a cardinal utility function, F, and an ordinal utility function, G. There are four bundles of goods, A, B, X, and Y. Written on one side, the back, of sheets one through four is the list of the elements that comprise bundles A, B, X, and Y, respectively. Using F, cardinal utility numbers 100 and 120 are generated for A and B, respectively. These numbers are written on the obverse sides of sheets one and two, respectively. Additionally, because 120 > 100, 2^{nd} and 1^{st} are also written on the obverse sides of sheets one and two respectively. Using G, an ordinal utility number, 2^{nd} and 1^{st} are generated for X and Y, respectively. These numbers are written on the obverse sides of sheets three and four, respectively.

Two additional bundles, C and Z, are now considered using two additional sheets of paper, five and six. Written on one side, the back, of sheets five and six is the list of elements that comprise bundles C and Z, respectively. Using F, a cardinal number, 110, is generated for C. This number is written on the obverse side of sheet five.

Then, using only the obverse sides of sheets one, two, and five, bundles A, B, and C can be cardinally ranked: B (120) > C (110) > A (100). Moreover, B is greater than C by 120-110 =10 or B is equal to (12/11)·C. Additionally B is greater than A by 120-100 = 20 or B is equal to (6/5)·A. And, C is greater than A by 110-100 = 10 or C is equal to (11/10)·A. Furthermore, A, B, and C can be ordinally ranked on the basis of their cardinal numbers: B – 1^{st}; C – 2^{nd}; and A – 3^{rd}. This last sentence might strike some readers as strange; after all, they might ask, how can it be valid to "ordinally rank" bundles on the basis of their *cardinal* numbers? Surely, it might be thought, ordinal rankings should be based upon *ordinal* numbers. However, for neoclassicists using utility functions, the ordinal ranking, B > C > A, arises from the cardinal ranking, U(B) = 120 > U(C) = 110 > U(A) = 100; i.e., the reason B is ranked first and assigned the ordinal number 1st is that its cardinal utility number is 120, which is greater than the cardinal utility numbers of C and A, 110 and 100, respectively. And, the reason that C is ranked second and assigned the ordinal number 2nd is that its cardinal utility number is 110, which is less than the cardinal utility number of B, 120, and greater than the cardinal utility number of A, 100. Lastly, the reason A is ranked third and assigned the ordinal number 3rd is that its cardinal utility number is 100, which is less than the cardinal utility numbers of both B and C, 120 and 110. That is, the ordinal ranking B >C >A and the ordinal numbers, B – 1st, C – 2nd, and A – 3rd, are derived from the cardinal ranking U(B) = 120 > U(C) = 110 > U(A) = 100. The point is that bundles A, B, and C can be ordinally ranked, but—for neoclassical economists—*the ordinal ranking is based on,* ***yea derived from****, their cardinal utility numbers.*

However, when we try to operate on bundle Z with G, we are stymied. We cannot operate on Z with G using only the information as to the elements that comprise Z and the obverses of sheets three and four. In fact, in order for G to generate an ordinal number for Z we must see the backs of sheets three and four, as well as that of sheet six.

The logical conclusion is that F is truly a cardinal utility function, and that any ranking of bundles based on such a function is cardinal in nature. Moreover, though an ordinal ranking can be generated from the cardinal ranking, the ranking is *essentially* cardinal in nature, with the ordinal ranking merely a by-product. Moreover, to use only the ordinal numbers generated as a by-product by cardinal utility functions is to fail to make use of the valuable information contained in the cardinal numbers that is not contained in the ordinal numbers.[48]

Furthermore, a truly ordinal utility function, such as G, cannot generate cardinal numbers or a cardinal ranking.

APPENDIX

Miller (1997, G-16); Ekelund and Tollison (1994, pp. 139-46); Frank and Bernanke (2001, pp. 106-11); and, Case and Fair (2002, pp. 111-13) are examples of principles level books, in which "utils" make an appearance. McCloskey (1982, pp. 41-48); Nicholson (2002, p. 78); and, Katz and Rosen (1991, pp. 54-60) are cases in point of intermediate level books in which "utils" are relied upon for purposes of economic analysis. Furthermore, it is

[48] Of course, such information is spurious; nevertheless, logically, if one uses a cardinal utility function, one should use it optimally, including using all of the useful information generated thereby.

interesting to note the following: (1) McCloskey (1982, p. 41), in discussing "an old fashioned way of looking at indifference curves and the consumer's choice that has been declared dead many times but refuses to stay in its coffin" uses "joys" (a unit of happiness) to measure happiness (utility), whereas in the section on indifference curves McCloskey (1982, p. 31) states that "his utility [happiness] would be 30 units;" note the absence of the prepositional phrase "of X," and particularly of the object of the preposition, X; i.e., McCloskey fails to tell us 30 units of *what*. (2) Katz and Rosen (1991 p. 56) state that "just because an ordinal utility function allows us to associate a certain number of utils to each bundle does not mean that we can objectively measure 'happiness.' Such an interpretation is ruled out because utils are only ordinal numbers." However, as noted above, units of necessity involve cardinality, not ordinality. (3) Nicholson (2002, p. 78) states, "that the units of utility measure (what we have, for lack of a better name, termed a util) drop out when constructing the MRS," as if the fact that the units drop out means they weren't there in the first place; but, again, units of necessity involve cardinality, not ordinality. (4) Frank (1991, pp. 91–95) and Varian (1992, p. 97) differentiate a utility function. They do so explicitly, with no reservations or apologies. It is difficult to see how this is possible if utility is purely an ordinal phenomenon.

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