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NOTES AND COMMENTS

Operators are *not* Parameters, the Dimensions of Operators and Variables *Must* be Invariant, and Indices may *not* be Dimensioned: Rejoinder to Professors Folsom and Gonzalez

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Abstract What is certain is that mathematics cannot possibly be a valid means (to advances in economic understanding) unless and until it is used properly. That means that dimensions must be used consistently and correctly. Barnett (Barnett, Quart J Austrian Econ, pp. 27–46, 2003) is about problems with the use of mathematics in economics involving the failure to use dimensions/units consistently and correctly. Professor Emeritus Folsom and Professor Gonzalez (Folsom and Gonzalez, Quart J Austrian Econ, pp. 45–65, 2005), hereinafter F&G, say, essentially, that what is correct therein is not new and that what is new is not correct. Additionally, they imply, by raising them, that I did not address issues that I should have, e.g., how to introduce dimensions into introductory economics and problems with the Cobb-Douglas (CD) function unrelated to dimensions. Herein, because of space limitations, I respond only to some of their criticisms. Responses to others are posted on the Ludwig von Mises Institute Working Papers site.

Keywords Operators · Dimensions of operators and variables · Indices · Mathematical economics

Operators are *not* Parameters

Barnett (p. 29) used a standard CD production function: $Q = AK^{\alpha}L^{\beta}$. Section one of F&G is hard to deal with because it confounds several issues; however, I think the key mistake they make, and which leads them into other errors, is that they

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misunderstand the nature of α and β in that function; to wit: F&G (p. 47, emphasis in original) state:[1]

> So far, so good—except for a minor quibble: we would describe A, α, and β all as "parameters." A quick-and-dirty definition of "parameter" might be "variable (changeable) constant"—a constant whose value (magnitude) can change, but only exogenously. Therefore, parameters are not *coordinate* variables.

Certainly, in this case, A is a parameter. But what of α and β? Are they parameters, also? They appear to be symbols for (real) numbers, but are they? In fact, they are neither parameters nor real numbers. Rather, in this context, both are *operator* names or symbols.[2]

One easy way to see this is to consider that $x^0=1\,\forall x$, such that $0^0=1$, whereas $0^x = 0\ \forall x \neq 0$. Obviously, the exponent 0 is not a number, but instead a function that maps every x → 1. And, that mapping applies to dimensions as well as magnitudes; for example, $10^0 \equiv (10\text{m})^0 \equiv \left((10\text{m})^2\right)^0 \equiv \left((10\text{m})^3\right)^0 \equiv 1$, where m = meter. Clearly, the exponent 0 is not the number 0, but an operator that maps any number, including dimensioned numbers, to one.

Furthermore, "An exponent is the power p in an expression of the form a^p. The process of performing the *operation* of raising a base to a given power is known as exponentiation" (emphasis added).[3] An equation of the form $y=a^x$ is a function, provided that for fractional values of x, the range is restricted to a single value for each value in the domain.[4]

It turns out, then, that F&G's "quibble" that α and β are parameters is not a quibble because it is not correct, and, unfortunately, improperly classifying α and β as parameters is not a trivial error, as will be shown below.

F&G (p. 48, italics in original, bold emphasis added) state:

> Therefore, a function's parameter dimensions (**and values**) simply are *whatever they need to be* (including roots and powers) to *describe the relationship fully and accurately*—including that the left and right side dimensions must match....
>
> *Variables* **must** have **(understandable)** dimensions. Parameters may or may not have dimensions. If they do not they are pure numbers (*not* necessarily invariant constants). If they do, their dimensions need not be understandable (although it is nice if they are). Instead, parameter dimensions' only requirement is that they *describe the relationship* between the function's dependent and independent variables' dimensions.

If one thinks that α and β are parameters, and as such may take on whatever values are necessary to "describe the relationship," and if one forgets that the operation of raising a variable to a power creates a function, it is easy to see how one

[1] In a somewhat lengthy footnote they explain the concept of a parameter.

[2] Perhaps I (Barnett, p. 29) misled F&G into this error by referring to α and β as "pure numbers," but I only intended to indicate that α and β, because they are exponents, could not be dimensioned.

[3] http://mathworld.wolfram.com/Exponent.html

[4] This may be accomplished by defining $a^x=|a^x|$; then $y=a^x$ meets the definition of a function provided ≥0. Thomas (1968, pp. 253–54)

can think F&G's conclusions that α and β may take on fractional values, and that the dimensions of A may be unreasonable.

Let us examine these conclusions. We may rewrite $Q = AK^{\alpha}L^{\beta}$ as Q = AVW = AZ, where $V = K^{\alpha}$, $W = L^{\beta}$, and Z = VW are functions and V, W, and Z are variables. Consider then the dimensions of the variables V and W. Because the dimensions of K and L are machine-hours/year and man-hours/year, respectively, the dimensions of $V(= K^{\alpha})$ and $W(= L^{\beta})$ make sense only if α=β=1. Any other value of α or of β yields meaningless dimensions; e.g., α=β=0.5 (yields machine-hours/year)$^{0.5}$ and man-hours/years)0.5. Moreover, even if α = b = 1, the dimensions of Z (machine-hours × man-hours)/year^2 are meaningless. Therefore, even if we accept arguendo, F&G's contention that parameters may take on any values that are necessary to "describe the relationship," α and β cannot take on fractional values, as they maintain, because that results in meaningless or, to use their term, unreasonable, dimensions.

Barnett (p. 30) also maintained that $\alpha = \beta = 1$ is unreasonable because: "For, if $\alpha = \beta = 1$, the marginal products of both K and L are positive constants (the law of diminishing returns is violated) and there are unreasonably large economies of scale—a doubling of both inputs, ceteris paribus, would quadruple output."

F&G (p. 48) maintain that "the solution is to require $\alpha < 1$ and $\beta < 1$," but note that "Barnett does not accept that solution." But as we have just seen, that "solution" violates the requirement, with which they agree, that variables have meaningful/reasonable dimensions.

F&G (p. 49) state:

> parameter A has dimensions [necessary] to describe ... [the] assumed relationship between output and input variables... to allow the ... α and β parameters to be the percentage change in output that results from a percentage change in real capital or labor input. So unless one rejects the idea that output quantity depends on input quantities, how can one reject the ideas that the percentage change in output quantity could depend on the percentage change of input quantity, and that the percentage relationship could be less than one? ... Barnett ... claims that any fractional values for the ... α and β parameters are "meaningless" or "economically unreasonable" simply because they are roots of input quantities. To us, that claim makes no sense, given the role of parameters in defining a production function's relationship between output and input.

Perhaps my claim that fractional values for α and β are meaningless or economically unreasonable would make more sense to them if they understood that α and β are operators, not parameters, and that input variables, and powers of such variables, include dimensions as well as magnitudes.

Moreover, that paragraph contains a serious *non sequitur*. Certainly inputs affect output and no doubt such relations may be expressed in terms of percentages, at least conceptually. One may agree with that, as I do, and still reject: (1) the idea that A may take on any dimensions necessary, however meaningless and unreasonable, so that α and β may be fractional elasticities; and, (2) that α and β may be fractions. All one needs do is remember that we are doing economics here after all, not mathematics, and reject the idea that the CD function can express economically meaningful and reasonable production relations.

To sum up my response to their first section, I maintain that: (1) their claim that α and β are parameters is incorrect—they are operators; (2) *a fortiori*, their claim that α and β may take on whatever values are necessary "to describe the relationship between the function's dependent and independent variables' dimensions" *because* they are parameters is incorrect; (3) their claim that variables must have understandable dimensions is correct, with the consequence that they are mistaken when they maintain that α and β may take on any values other than one;[5] and, (4) *a fortiori*, their claim that because α and β are elasticities, they may take on fractional values is incorrect.

Dimensions Must Be Constant

Barnett (pp. 31–32) analogized between the CD production function and the Newtonian gravitational function. Specifically, it compared the dimensional constancy of the gravitational constant, *G*, in the latter with the dimensional inconstancy of the technology parameter, *A*, in the former. Whereas *G* necessarily has dimensions of length^3/(mass·elapsed-time^2; e.g., m^3/(kg·s^2, *A* has the units of $Q/\left(K^{\alpha}L^{\beta}\right)$, which in my example are $(\text{widgets/year})\Big/\left((\text{machine}-\text{hours/year})^{\alpha}(\text{man}-\text{hours/year})^{\beta}\right)$. Because neither the value of α nor that of β is theoretically given for any specific production relation, they must be estimated. Different estimates can and do yield different values. Consequently, even if the dimensions of the variables *Q*, *K*, and *L* are constant, the dimensions of the variables K^{α} and L^{β} are not. A necessary result of this is that the dimensions of *A* also vary with α and β. F&G apparently think that the explanation for this difference (constant dimensions for *G*; variable dimensions for *A*) is that in the equation for *G* there are "*no variable parameters* [emphasis theirs]" on the right side whereas (they believe, incorrectly) in the equation for *A* there are variable parameters in the form of α and β (again, note that neither α nor β is a parameter). Perhaps, had I spelled out the intermediary step that when the values of the operators α and/or β vary the values of the dimensions of variables K^{α} and L^{β}, respectively, change causing the dimensions of *A* to vary, also, they might have understood why the dimensions of *A* must be constant for any particular production relation.

Furthermore, F&G (p. 50, emphases in original) state:

> Actually, however, for all values of α and β, the *dimensions* of *A* are invariant, and remain defined by (1a) or (1b) above (emphasis in original). The exponent on "yr" (elapsed time) *always* is $\alpha+\beta-1$, the exponent on "caphr" (machine-hours) *always* is α, and the exponent on "manhr" (labor-hours) *always* is β. Different values of α and β change *only the magnitude* of *A*.

However, F&G are incorrect on both points: in fact, contra F&G, the *dimensions* of A *must* be affected by a change in α and/or β, whereas the *magnitude* of *A must*

[5] It is, however, possible that in different CD production function exponents could take on, in addition to 1, the integer values 2 or 3 if the relevant input was some good appropriately measured in, say, m^2 or m^3 respectively, although in such case the function would no longer exhibit diminishing returns and the economies of scale would be unrealistic; i.e., its economic properties would not be acceptable.

not be. Because $Q = AK^{\alpha}L^{\beta}$ is a dimensioned equation, in which each element, save for the operators α and β, has both magnitude and dimensions, it will help to rewrite it making explicit the magnitude and dimensions of each element, save for the operators, by writing the magnitudes in lower case and the dimensions in brackets, such that; e.g., q = magnitude of Q and $[Q]$ = dimensions of Q. Then $q[Q] = a[A](k[K])^{\alpha}(1[L])^{\beta}$. We may separate this into two equations, one involving only magnitudes and the other only dimensions. They are, respectively: $q = ak^{\alpha}1^{\beta}$, and $[Q] = [A][K]^{\alpha}[L]^{\beta}$.

Consider, first, F&G's claim that the dimensions of A are not affected by a change in α and/or β. Isolating $[A]$ in the dimensions-only equation yields $[A] = [Q]/\left([K]^{\alpha}[L]^{\beta}\right.$, where $[Q]$, $[K]$, and $[L]$ are known and *constant.* There is no reason for any or all of $[Q]$, $[K]$, and $[L]$ to change because of a change in α and/or β. Therefore, any change in α or β *must* change $[A]$; i.e., the dimensions of A.[6] Q.E.D.

Next, re the magnitude of A, a, because the magnitudes of α $(=\{\partial q/q\}/\{\partial k/k\}$ and β $(=\{\partial q/q\}/\{\partial l/l\}$are independent of a, and because in this simplistic model of production there is no time lag between when the inputs enter the process and the output emerges, a is independent of the magnitudes of α and β, also. Therefore, there is no reason for a to change *because* α and/or β changes.[7] Q.E.D.

Moreover, F&G's positions are problematical even if we assume, arguendo, both that α could change and that such a change would affect a. Consider that $q = ak^{\alpha}l^{\beta} \Rightarrow a = q/\left(k^{\alpha}l^{\beta}\right)$. Then $\partial a/\partial\alpha = -(q \cdot \ln \mathrm{k})/\left(\mathrm{k}^{\alpha}1^{\beta}\right) = -a \cdot \ln \mathrm{k} < 0$;[8] i.e., ceteris paribus, there is an inverse relation between A, the technological factor, and α.[9] Therefore, if an increase in the output elasticity, α, affects the magnitude of the technology parameter, a, it does so in a negative way; i.e., it decreases it. This is unreasonable and therefore we must conclude that a change in α has no effect on a.[10]

F&G (p. 50) then offer this:

> In the gravity model and in the Cobb–Douglas production function model, the estimations or measurements are very different conceptually. In the gravity model, regardless whether its left-side variable is F or G, all symbols other than G are known values; the only unknown is G. In the Cobb–Douglas model, the magnitudes of A, α, and β all are unknown, and are estimated as a "best fit" to

[6] Note that because $[K]$ and $[L]$ differ, it is impossible for a change in α to be offset by a change in β, and *vice versa*.

[7] It is important to keep in mind in this discussion of the effects of changes in α and/or β, that such changes are illogical in that the dimensions of the inputs must be meaningful and economically reasonable. Consequently, it is virtually impossible to think of any case in which an input in a CD function could have more than one valid exponent, and, therefore, for which a change thereof would be permissible.

[8] The domain of k is: $k \geq 0$; if $k=0$, $\ln k$ is undefined and, *a fortiori*, so is $a\ln k = \partial a/\partial\alpha = 0$. $\mathrm{k} = 1 \Rightarrow \ln\mathrm{k} = 0$ and $a\ln k = \partial a/\partial\alpha$. $0<k<1$ is unreasonable as it means that the capital input is less than one machine-hour per year and there is only one other input, labor.

[9] Because I stressed the point, supra, that exponents are operators, not parameters, the reader might well inquire if it is legitimate to take the derivative of a variable with respect to an operator. It is if the operator is a variable; e.g., one may take the derivative of $y=e^x$ with respect to x, where is the base of the natural logarithms (technically, noninteger exponential operators are functions only with proper restrictions. See footnote 4, supra).

[10] The same analysis applies, *mutatis mutandis*, for the relation between β and a.

> known data for output quantity q and for input quantities K and L. And while the gravity model applies to the theoretically identical quantitative behavior of objects in one universe, the Cobb-Douglas model ... has been used ... to describe the behavior of widely disparate enterprises: single firms in entirely different industries (peanut farming, ... internet services, ... pharmaceuticals invention and production; nonprofit private organizations; local, state, and national governments—and price-weighted aggregate data for multi-output enterprises, entire industries, and even entire economies)[.

There are difficulties here. First, their claim that "all symbols other than *G* are known values," is misleading. The dimensions of the various variables (mass, distance, and force) in the Newtonian gravitational equation determine the dimensions of *G*. That said, scientists measure the magnitude of *G* experimentally. But they know before, during, and after each experiment, the dimensions of *G*; to wit: $m^3/(kg{\cdot}s^2)$.[11] I do think that a serious problem for mathematical economics and econometrics is that, save in rare cases,[12] we do not know the exponential operators or the dimensions of our variables and/or parameters, before, during, or after our measurements. For instance, for labor inputs into a production function, one would probably need almost as many dimensions as there are workers; in fact we might need more dimensions than there are workers just for the labor input. For example, one dimension might be a Bill-Barnett-hour-on-a-good-day-for-him; another might be a Bill-Barnett-hour-on-a-middling-day-for-him; and, a third might be a Bill-Barnett-hour-on-a-poor-day-for-him.

F&G (p. 51, emphases in original) continue to try to make a case that it is reasonable for A, α, and β to vary: "But in specific economic functions, including production functions, parameters such as A, α, and β are *not* defined invariant constants: we *measure* them as *variable* parameters that *of course* change to fit different situations and different time periods, as they are *expected* and *supposed* to do." But, this is to no avail. Why, pray tell, are the dimensions of an input "*expected*" and "*supposed*" to *change*?[13] Is an unskilled man-hour today different than last year's unskilled man-hour? Even more awkward, what happens when the estimated exponent of the electricity input[14] (measured in, say, kilowatt-hours) of a production process varies. Suppose, e.g., the electricity input, E, to be composed of magnitude, e, and dimensions/units, kilowatt-hour. Then, if we enter this term, appropriately into a (more generalized) CD function, and its exponent is γ, the function becomes: $Q = AK^{\alpha}L^{\beta}E^{\gamma}$. So, then, for a given process, if e=100, and if different measurements of γ yield γ=1 and γ=2, then are we to say that the electricity input is E=100 kilowatt-hours at one time and E=10,000(kilowatt-hours)^2

[11] Physics is an experimental science in the sense that regardless of how they originated, the laws of physics are not accepted unless they have been empirically verified.

[12] An example is the equation of exchange, in disaggregated form; the aggregated version involves absurd variables: the quantity of some composite good (as if we can meaningfully sum quantities of heterogeneous goods), as well as its average price.

[13] This is not to deny that differing production processes use differing resources (at least in part) and thus that the dimensions of the differing resources necessarily are different. Nevertheless, the *dimensions*, in contradistinction to the magnitude, of a specific input in a *particular* process must be constant.

[14] Certainly virtually any production process that uses electricity should include an energy input.

at another? Surely, this is mistaken (it is no less so if we use fractional exponents in keeping with diminishing returns).

F&G (p. 51) persist:

> Specific production functions are not part of the realm of pure economic theory, but are tools of historical analysis. To demand constancy for a production function's parameter values simply makes no economic sense.
>
> Professor Barnett's analysis has not shown us any dimensional errors in the Cobb-Douglas production function.

Again, they do not understand that α and β are not parameters but, rather operators *operating on dimensions* (as well as magnitudes), and therefore cannot take on any values "to fit different situations and different time periods."

They are correct when they say (emphasis added): "To demand constancy for a production function's parameter *values* simply makes no sense." But that is irrelevant to the case in point, as the issue is the constancy of the *operators* and, therefore, the constancy of the dimensions of the input variables and technology parameter in the CD production function. And it most assuredly does make sense to demand that these dimensions be constant, not to mention meaningful. What does not make sense is variable dimensions.

F&G (p. 50, n.5) quote Jong:[15]

> De Jong writes (1967, p. 19, n.1): "The dimension of a certain variable tells us how the numerical value of that variable changes when the units of measurement are subjected to change." The same statement would apply to parameters.

These authors misunderstand this statement of Jong. Of course, Jong's statement applies to parameters as well as variables. But that is irrelevant to the issue Jong is concerned with and also to the issues at hand. Jong is making the point that if we change the *units* (in contradistinction to the *dimensions*) in which a particular variable is measured, then the magnitude of the variable must change also, and that in a fixed way. If, e.g., the variable we are interested in is the distance between two points, and we the use meter, m, as the measure of the length dimension, then a particular distance, *X*, may be, say, 1,000 m; i.e., it has magnitude 1,000 and its unit is *m*. If we then change the unit in which we measure the length dimension from m to kilometer (km), where 1 km=1,000 m, then the magnitude necessarily changes to 1; i.e., the variable's magnitude now equals 1 and its unit becomes the km. And, these are identical; i.e., 1000 m≡1 km.[16] Of course, the dimension of the variable, length, does not change.

However, that has nothing to do with the case in point. It is unrelated to the issues I was considering. My point was that the relevant *dimensions* must not change. In reality, although the units in which people measure particular variables sometimes differ, the *dimensions* never change. We may measure a distance in m or km or mi,

[15] Although F&G refer to Frits J. DE Jong as De Jong, I refer to him as Jong, as his name is listed as Jong, F. J. DE in the index of names in his book, *Dimensional Analysis for Economists*.

[16] This is obvious from Jong's (1967, pp. 18–19) example where he compares the magnitudes that result from when the national income of the Netherlands in "millards guilders per annum" and then in "'tons' per month" by "substituting 'tons' for guilders ... and months for years ....", where "1 guilder=1/100,000 ton ..."

etc., but we cannot measure that same variable in m^2 or km^3 or mi^4, or kg or sec or any other non-distance unit. And that is precisely one of the issues I raised. That is we cannot measure the K input as K^α, with, say, α=1/2 for one set of measurements and α=1/3 for another, as that changes the dimensions in which the variable is measured; it does not maintain the dimensions while merely changing the units in which the dimension is measured. For example, if K=64 machine-hours/year, then if α=1/3, $K^\alpha = 4$ (machine-hours/year)$^{1/3}$, but if α=1/2, $K^\alpha = 8$ (machine-hours/year)$^{1/2}$. Ignoring the effect of the different αs on the magnitude, (machine-hours/year)$^{1/3}$ is an entirely different animal than (machine-hours/year)$^{1/2}$ (of course, neither of these is understandable, much less economically meaningful or reasonable).

F&G (p. 51, n.7) states:

> For a much more formal and "philosophical" analysis of Cobb-Douglas (and constant elasticity of substitution) type functions, see Jong (1967, pp. 34–50; for noninteger exponents, pp. 46–50). Our discussion of the Cobb-Douglas function has treated it as a "fundamental equation," analyzed using the "traditional method" (pp. 34–37).

As to Jong's (1967, pp. 46–50) analysis of the CD function insofar as dimensions are concerned, he (Jong 1967, pp. 46–47, emphasis and footnotes added) states:

> Unfortunately, however, *dimensional* formulae whose exponents are not integers carry one serious difficulty with them: as has been pointed out in the Mathematical Appendix, dimensional formulae of this type do not satisfy the requirements imposed by the theory of the algebraic structure of dimensional analysis. Indeed, hitherto no algebraic theory has been formulated which permits defining A^α for values of α which are not integers in a way that is satisfactory from a mathematical point of view. This can only be done for integral values of α.
>
> The consequence of this is rather serious: it means that it is *illegitimate* to write N^α, $K_a^{1-\alpha}$ or K_a^α, u^α in the two production functions considered in sections 8.1 and 8.2, since it is *impossible* to define these "quantities."[17] In other words, these "quantities" are *meaningless*. The same objection applies to the dimensional[18] constants that occur in these production functions, as becomes obvious from inspecting their dimensional formulae where $0 < |\alpha| < 1$:
>
> $$c_a = \left[R_e R_a^{-\alpha} R_K^{\alpha-1} T^{\alpha-1}\right]$$
>
> [...][19]

There are two ways out of this difficulty.

Unfortunately each of the two methods Jong (1967, p. 47) suggests as ways to resolve this problem are unsatisfactory. First, he suggests "that one treats directly the constants and variables stripped of their units of measurements (the latter must be

[17] These are the CD and CES functions that Jong writes as: $u = c_a N^\alpha K_a^{1-\alpha}$ and $u = \left(c_{aN} N^\alpha + c_{aK} K_a^\alpha\right)^{1/\alpha}$,

[18] The "dimensional constants" Jong refers to are the c_a in his CD production function that is the counterpart of A in our CD function, and the c_{aN} and c_{aK} is his CES function.

[19] In Jong (1967) T inside of bracket is the time dimension and the R_i, i=1...n, in side of brackets are n real dimensions; i.e., the dimensions of n goods.

specified in the verbal text)" and claims that "this interpretation is entirely legitimate." That is, he completely ignores the problem. Although this approach to dimensions is ubiquitous in economics, I maintain that it is entirely invalid as it denies, de facto, the reality that most economic variables have dimensions.[20] Jong's second method is to eliminate the dimensions of the inputs by measuring them as ratios. But this method is problematical, also.

We see, then, that Jong recognizes the problem generated when the exponents of (operators on) the input variables in the CD function are fractions, even though his attempts to elude it are unsuccessful.

Indices do *not* have Dimensions

F&G (p. 62, appendix B) revert to the matter of dimensions/units, now in regard to indices, using Fisher's equation of exchange to illustrate their position.[21]The issue of dimensioned indices arises in their critique my "Macroeconomic Example" (Barnett, pp. 32–34), in which they state: "In general, *an index need not be dimensionless*" (F&G, p. 56, emphasis in original). As authority for that proposition they cite Jong (1967, pp. 22–23, emphasis in original): "Are not index numbers dimensionless products? The answer is: this *may* well be so, but not necessarily. *The answer 'yes' or 'no' depends upon what is best adapted to the problem at hand.*"

Examination of F&G's example greatly strengthens my key point; to wit: that if economists are to use mathematics, they should use dimensions/units consistently and correctly.

The units of MV in the equation of exchange, for both the transactions and the nominal income versions are dollars/year.[22] (F&G, p. 63)

In consequence of that, F&G (pp. 63–64, emphasis in original) then state:

> Therefore, in the equation of exchange [MV=PT], price index P as a dimensionless pure number ratio and transactions quantity index T as a dimensionless pure

[20] One may hypothesize that in many cases this is done in order to further the pretense of rigorous mathematical analysis on the part of those economists suffering from "physics envy." (I do not aim this criticism at F&G.) Interestingly, given the failure to use dimensions consistently and correctly in economics, the term "mathematics envy" would be more appropriate than "physics envy."

[21] F&G (p. 63, n.31) note that their work on the equation of exchange is: "Based on Boulding (1966, pp. 27–28); and loosely on Jong (1967, pp. 23–30)." Re Boulding, this is surprising as the relevant section on those pages is titled: "Consistency of Indices." And that is precisely what is dealt with there. He constructs: $V_0 = \sum p_0 q_0$; $W_0 = \sum p_1 q_0$; $W_1 = \sum p_0 q_1$; and $v_1 = \sum p_1 q_1$. Boulding then says: "From these we derive four possible concepts of price and quantity indices, or relatives for the end date with the base-date index=1." Note that Boulding, himself, refers to the indices as "relatives," and fixes the base-date index = [dimensionless] 1. He then derives a set of four indices, comprising two Laspeyres' and two Paasche's, one each for prices and quantities. He then notes that the only products of one price and one quantity index that are mathematically consistent are those where one of the indices is a Laspeyres" and the other a Paasche's. He concludes that section: "The rule, therefore, emerges that if end-date price weights are used in calculating the quantity index, base-date quantity weights must be used in calculating the price index, and vice versa." There is absolutely nothing in the section cited by F&G that could in any way lend support to their concept of a dimensioned index.

[22] F&G (p. 64) mention the problem of determining the "point in time" relevant to determination of the money stock, but as that affects only the magnitude, and not the dimensions/units, of M and MV, it irrelevant to this discussion.

> number ratio, *cannot hold simultaneously.* The solution is that either one or the other ratio must be multiplied by the flow of nominal transaction quantities during the base time period, thus making either a P or T dimensioned index.
>
> That conclusion applies not only to Fisher's equation (MV=PT), but also to a "nominal income" equation of exchange (MV=PQ), in which the right side does not include intermediate transactions but instead is nominal Gross Domestic (or National) product [Q or sometimes Y being *real* Gross Domestic (or National) product], and Velocity is a substantially smaller number than in Fisher's equation....
>
> On Fisher's right side, P is a price index, and T is a transactions quantity index. Let p_j and q_j represent the price and quantity of the j-th transaction, and superscripts ^{b} and ^{e} respectively represent the base period and the end period. Assume P is a Laspeyres's index (base period quantity weights). Then the end-period dollar value of the flow of transactions receipts per time period is $PT = \sum p_j^e q_j^e$, where
>
> $$P = \sum p_j^e q_j^b \Big/ \sum p_j^b q_j^b, \text{ and } T = \left(\sum p_j^e q_j^e \Big/ \sum p_j^e q_j^b\right) \sum p_j^b q_j^b.$$
>
> The transaction index T, rather than a pure ratio, has been redefined as a dimensioned index.

This is highly problematic. Of course, both Laspeyres's and Paasche's indices are dimensionless. Are we, then, to understand that F&G think that the multiplication of a dimensionless Paasche's quantity index by the dimensioned variable (beginning-period-dollar-flow of transactions receipts) results in a product that is a dimensioned index in contradistinction to a dimensioned variable?

Second, consider the genesis of F&G's "T." The ratio of end-of-period expenditures to those of the beginning of period $\left(\sum p_j^e q_j^e \Big/ \sum p_j^b q_j^b\right)$ may be expressed as the product of a Laspeyres' and a Paasche's indices.[23] Let P^e and T^e be the Laspeyres' price index $\left(\sum p_j^e q_j^b \Big/ \sum p_j^b q_j^b\right)$, and Paasche's quantity indices $\left(\sum p_j^e q_j^e \Big/ \sum p_j^e q_j^b\right)$, respectively. Then $P^e T^e = \sum p_j^e q_j^e \Big/ \sum p_j^b q_j^b$. Note that F&G's $PT = \sum p_j^e q_j^e$. Therefore $P^e T^e = PT \Big/ \sum p_j^b q_j^b$ or $PT = P^e T^e \sum p_j^b q_j^b$. By separating the right side into two terms P^e and $T^e \sum p_j^b q_j^b$ the dimensions of $\sum p_j^b q_j^b$ (dollars/year) are arbitrarily attached to the Paasche's quantity index. And that, according to F&G, converts the dimensionless Paasche's quantity index into a "dimensioned (quantity) index" of some sort.

F&G realize that the price index could be redefined as a Paasche's index instead of a Laspeyres's index, mutatis mutandis. They also realize that the $\sum p_j^b q_j^b$ could be "attached" to the price index instead of the transactions index, so that the dimensioned index would be the price index, not the transactions index. Specifically, they state (F&G, p. 64):

> The transactions index T, rather than being a pure ratio, has been redefined into a dimensioned index.

[23] To see this, multiply the ratio by one in the form of $\sum p_j^e q_j^b \Big/ \sum p_j^e q_j^b$ and rearrange terms appropriately.

Alternatively, the Laspeyres's price index P could be redefined into a dimensioned index as $P = \left[\sum p_j^e q_j^b / \sum p_j^b q_j^b\right] \cdot \sum p_j^b q_j^b = \sum p_j^e q_j^b$, so that T would be a pure ratio.

Analogous results hold if P is a Paasche's price index (end period quantity weights), and T is defined accordingly (the final factor of either Tor P again is $\sum p_j^b q_j^b$).

Certainly, they should have realized something was wrong with their concept of dimensioned indices when they saw that they could "attach" $\sum p_j^b q_j^b$ to either of the other two terms, each of which was a dimensionless index to create their dimensioned indices.

That is, any of the following will do for them.

A Laspeyres's price index with a dimensioned transactions index:

$$P^e = \left(\sum p_j^e q_j^b / \sum p_j^b q_j^b\right) \text{ and } T^e = \left(\sum p_j^e q_j^e / \sum p_j^e q_j^b\right) \times \sum p_j^b q_j^b;$$

or, in terms of dimensions, in brackets :

$$P^e = [\$/\text{year}]/[\$/\text{year}] = [1]; \text{ i.e., dimensionless}$$
$$T^e = ([\$/\text{year}]/[\$/\text{year}]) \times [\$/\text{year}] = [\$/\text{year}]$$

A Paasche's price index with a dimensioned transactions index:

$$P^e = \left(\sum p_j^e q_j^e / \sum p_j^b q_j^e\right) \text{ and } T^e = \left(\sum p_j^b q_j^e / \sum p_j^b q_j^b\right) \cdot \sum p_j^b q_j^b;$$

or, in terms of dimensions, in brackets :

$$P^e = [\$/\text{year}]/[\$/\text{year}] = [1]; \text{ i.e., dimensionless}$$
$$T^e = ([\$/\text{year}]/[\$/\text{year}]) \cdot [\$/\text{year}] = [\$/\text{year}]$$

A dimensioned price index with a Laspeyres's transactions index:

$$P^e = \left(\sum p_j^e q_j^e / \sum p_j^b q_j^e\right) \cdot \sum p_j^b q_j^b \text{ and } T^e = \sum p_j^b q_j^e / \sum p_j^b q_j^b;$$

or, in terms of dimensions, in brackets :

$$P^e = ([\$/\text{year}]/[\$/\text{year}]) \cdot [\$/\text{year}] = [\$/\text{year}]$$
$$T^e = [\$/\text{year}]/[\$/\text{year}] = [1]; \text{ i.e., dimensionless}$$

A dimensioned price index with a Paasche's transactions index:

$$P^e = \left(\sum p_j^e q_j^b / \sum p_j^b q_j^b\right) \cdot \sum p_j^b q_j^b\Big) \text{ and } T^e = \left(\sum p_j^e q_j^e / \sum p_j^e q_j^b\right);$$

or, in terms of dimensions, in brackets :

$$P^e = ([\$/\text{year}]/[\$/\text{year}]) \cdot [\$/\text{year}] = [\$/\text{year}]$$
$$T^e = [\$/\text{year}]/[\$/\text{year}] = [1]; \text{ i.e., dimensionless}$$

Note, first, that the dimensions of F&G's dimensioned transactions index are anomalous, in that, where one would expect them to be transactions per year, they are, instead, dollars per year. And, that the dimensions of their dimensioned price index are anomalous, also, in that, where one would expect them to be dollars per unit of a "composite good," they are, instead dollars per year.

Note, next, that the dimensions of F&G's dimensioned transactions index are identical to the dimensions of their dimensioned price index.

One has to think that in a paper centered on the issue of the failure to use dimensions consistently and correctly in economics, that had F&G themselves paid attention to dimensions in their dimensioned index formulation of Fisher's equation of exchange, they would have realized the errors involved. That is, certainly they would have come to appreciate that the dimensions of their indices were, at one and the same time, incorrect and identical.

In sum, Professor's F&G's "dimensioned indices" are no such thing.

Summary

First, F&G are incorrect when they claim that the exponents, α and β, in the CD function are parameters. Consequently and second, F&G are incorrect when they claim that α and β in the CD function need not be constant because they are parameters, when in fact they are operators, not parameters. Third, because the dimensions of the parameter A in the CD function are a function of the dimensions of the variables, Q, K^{α}, and L^{β}, that necessarily are constant and meaningful/reasonable, F&G are incorrect when they maintain that the dimensions of A may vary and/or be meaningless/unreasonable. Third, F&G are incorrect when they maintain that indices may be dimensioned.

Conclusion

Barnett (36) concludes: "What is certain is that mathematics cannot possibly be a valid means [to advances in economic understanding] unless and until it is used properly. Among other things, that means that dimensions must be used consistently and correctly." F&G's paper provides no reason to alter it in any way; in fact, because of errors re dimensions made therein, it reinforces that conclusion.

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References

Barnett, William II. 2003. "Dimensions and economics: some problems" *The Quarterly Journal of Austrian Economics*. 6 (3): 27–46. http://www.qjae.org/journals/qjae/pdf/qjae6_3_2.pdf. Production errors rendered the hard copy unusable; the url site has the correct version. Translated and reprinted as: "Wymiary a ekonomia; niektóre problemy." *Studia Ekonomiczne*. 3 (L): 187–205.

Boulding, Kenneth. 1966. *Economic analysis: vol. II—macroeconomics*. 4th ed. Harper & Row, New York.

Folsom, Roger Nils, and Rodolfo Alejo Gonzalez. 2005. "Dimensions and economics: some answers." *Quarterly Journal of Austrian Economics*. 8 (4): 45–65.

Jong, Frits J. DE. 1967. *Dimensional analysis for economists*. North-Holland, Amsterdam.

Thomas, George B., Jr. 1968. *Calculus and analytical geometry*. Addison-Wesley, New Jersey.