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NOTES AND COMMENTS

Cantor's Diagonal Argument: An Extension to the Socialist Calculation Debate—A Comment

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Abstract In a recent article Robert P. Murphy (2006) uses Cantor's diagonal argument to prove that market socialism could not function, since it would be impossible for the Central Planning Board to complete a list containing all conceivable goods (or prices for them). In the present paper we argue that Murphy is not only wrong in claiming that the number of goods included in the list should be uncountable, but also that the number of equations/prices is irrelevant from the point of view of market socialism.

Keywords Argument · Calculation · Debate

In a recent paper Robert P. Murphy suggested a new approach to defend Ludwig von Mises's famous argument that the socialist economy is not only inefficient, but quite simply impossible (Murphy 2006). In his paper Murphy first gives a brief account of the socialist calculation debate and then focuses on the mathematical treatment of the calculation problem, the general idea of which is that a fully rational allocation of resources could *theoretically* be obtained under socialism as a solution of a carefully devised system of equations. As originally proposed by Barone (1935) and later elaborated and modified by Dickinson (1933) and supposedly Lange (1936), the mathematical method had seemingly but one flaw: the number of equations describing the socialist market would be unimaginably huge. In fact, as Murphy points out—and this is where the novelty of his approach is revealed—the number of such equations, or equivalently prices resulting from them, would be not only

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infinite but also uncountable. After reaching this point Murphy uses Georg Cantor's famous diagonal argument to prove that an uncountable number of equations cannot be solved for the simple reason that a list of such equations could not even be constructed, and hence the sought after vector of prices could not be obtained.

In the present paper we argue that Murphy is wrong in claiming that the set of equations (prices) describing the socialist economy is uncountable. We also show that even if—as we believe is the case—the number of such equations is countable this conclusion has no bearing on the calculation debate, for it is insignificant from the point of view of the economy as such. We should also mention that although the case of countability is of no importance in a realistic approach to economics, it is, however, of great significance for axioms used throughout neoclassical economics.

The (Un)countability of the Set of Equations

Let us first recall the basic definitions in case the readers are not familiar with the notions of countable and uncountable sets. For the purpose of this paper we need not go into technical details of what a set is, and we rely instead on basic intuition.[1] A set is called countable infinite if it has many elements such as the set of natural numbers or, equivalently, if it can be arranged into a list (if there exists a one-to-one correspondence between our set and the set of positive integers). The set is called uncountable (and infinite) if it has "more" elements than the set of natural numbers, meaning that it cannot be put into a list (formally, if there does not exist a one-to-one correspondence between the set and positive integers). As Murphy notes, an example of a countable set is the set of all even integers $\{2, 4, 6, \ldots, 2n, \ldots\}$ and an example of an uncountable set is—by Cantor's diagonal argument—the interval [0,1].

The readers should however bear in mind that although we have just provided an example of an uncountable set, such objects cannot be meaningfully said to exist in the real, physical world. They are but abstract concepts and make sense only in the rigorous realm of modern mathematics. Somewhat ironically, this point has been observed even by Robert Aumann, the very person who incorporated the concept into economic reasoning. Writes Aumann (1964, pp. 39, 41):

> We submit that the most natural model for this purpose [of developing the notion of perfect competition] contains a *continuum* of participants, similar to the continuum of points on a line or the continuum of particles in a fluid.... The idea of a continuum of traders might seem outlandish to the reader. Actually, it is no stranger than a continuum of prices or of strategies or a continuum of "particles" in fluid mechanics. In all these cases, the continuum can be considered an approximation to the "true" situation in which there is a large but finite number of particles (or traders or strategies or possible prices). (emphasis added).

This short note is not the place to discuss the use—or misuse—of continuum in economics, but the following quote from Paul Ormerod gives the right impression on

[1] Readers interested in a much more detailed treatment of the basic set-theoretic concepts and theorems should consult any book on set theory, e.g., Hrbacek and Jech (1999) or a fairly advanced treatment in Jech (2002).

how absurd it would be to try to use uncountable sets in economics (or social sciences in general):

> To take just one example, the phrase "assume a continuum of traders" will be encountered in many theoretical papers on the idealized market economy.... But what does this phrase "continuum" actually mean? It sounds quite innocuous, yet spelt out in words it might lead people to query the realism of any academic paper based on this assumption, or even begin to doubt whether the article was worth writing in the first place. For the phrase means that the number of people, whether as individuals or as firms, carrying out trade in this theoretical economy is not just large but quite literally infinite. In fact, to be strictly accurate, it even means rather more than this. If one were to start to count the whole numbers—one, two, three and so on—one could go on for ever. There is an infinite number of them.... But mathematicians have the apparently bizarre but nevertheless logical concept of infinities which are even bigger than this infinity! A continuum is exactly one of these. In other words, it is assumed that there is not just an infinite number of traders, in the sense that the set of whole numbers is infinite, but there is an even bigger number of them than this. (Ormerod 1995, pp. 43–44).

After these preliminary remarks we may move on to the question of what do the countable and uncountable sets have to do with the calculation debate. Murphy argues that the principal objection to Mises's claim that socialism could not calculate, and hence effectively allocate resources, was the mathematical argument that given consumer preferences and all the relevant information concerning production functions, etc., the omnipotent dictator could arrange a list of equations determining the equilibrium, market clearing prices. Should the conditions change in any significant way, the planner would simply tinker with the original vector of prices and by a method of trial and error reestablish another temporary equilibrium. Thus, so long as computational power is not the issue, the socialist economy could function just as smoothly as the free market. Strictly speaking, however, this was the contention of Barone and Dickinson rather than of Lange himself. In his famed essay Lange clearly explains that his answer to Mises has nothing to do with equation-solving:

> Thus the accounting prices in the socialist economy can be determined by the same process of trial and error by which prices on the competitive market are determined. To determine the prices the Central Planning Board does not need to have "complete lists of the different quantities of all commodities which would be bought at any possible combination of prices of the different commodities which might be available" (Hayek 1935, p. 211). Neither would the Central Planning Board have to solve hundreds of thousands (as Professor Hayek expects) or millions (as Professor Robbins thinks) of equations. The only "equations" which would have to be "solved" would be those of the consumers and the managers of production plants. These are the same 'equations' which are solved in the present economic system.... Consumers "solve" them by spending their income so as to get out of it the maximum total utility (Lange 1936, pp. 66–67).

And that is exactly why Hayek in response to Lange did not return to the equation-solving argument, although he mentions that this sort of mathematical solution has been proposed at some point (Hayek 1940, pp. 127–28). But leaving this aside, Murphy's argument runs as follows:

> if the socialist planners really are to mimic the market outcome, they would need to publish a list containing, not merely a huge number of prices, and not merely an infinite number of prices, but rather a list containing an uncountably infinite number of prices. (Murphy 2006, p. 8)

To clarify why this should be so Murphy considers the case of innovation:

> Back when Hayek and Lange were arguing, there were no market prices for, say, laptop computers. Thus, even if socialism could mimic the market for a few years, it would eventually fall short because it would lack the introduction of new goods that is so typical in market economies. (Murphy 2006, p. 8)

Nevertheless, this statement seems to contradict what Lange thought about his own method, since he was not at all concerned about goods that would be invented and produced only at some later point in time. We also fail to see why the fact of not having accounted for all the possible new inventions should pose any serious problem to Lange's solution. But for the sake of argument let us suppose that this is so (we shall come back to this assumption later on) and that socialism really should fall short after not having taken into account the introduction of new goods. Murphy claims that the straightforward answer that a defender of Lange's method would give to that charge is that all the Central Planning Board needs to do is include in its calculations all the conceivable goods and then arbitrarily set prices for them high enough so as to make sure that, since the quantity supplied is 0 (goods are not being produced yet), the quantity demanded would also be 0. But then follows a *non sequitur* on Murphy's part. He states that:

> Once we realize that all conceivable goods and services that might be offered, must have corresponding prices included in the planners' official lists, we understand that such lists would necessarily contain an uncountably infinite number of items. (p. 9)

According to Murphy this "is inescapable," but all he offers to support his claim is that the planner would have to publish prices, e.g., for every single book that could possibly be written in the future. Fair enough, but how does this prove that the number of equations (prices) set up by the planning committee should be uncountable?[2] Murphy seems to believe that the number of books that can

[2] We note as an aside that it is quite unclear what the latter exactly means. It is one thing to speak about infinity in the realm of mathematics, where it is given a rigorous explanation, and it is quite another to use the concept in the domain of social sciences and everyday life, where it is bound to breed nothing but confusion. On the general trend of importing mathematical (and scientific) terminology to social sciences see in particular Andreski (1972) and Sokal and Bricmont (1999).

conceivably be written is uncountable, but there is nothing further from the truth. As the linguist Stephan Anderson puts it:

> A human language such as English…produces new messages…by combining discrete elements in new ways. As a result, the number of sentences in such a system is the same as the number of rational numbers, or $_0$. (Anderson 2004, p.79)

What Anderson implies is that the set of all conceivable sentences in English, or any other language for that matter, is countable. Now, any book may be thought of as a finite sequence of sentences (maybe repeating themselves). An elementary set-theoretic result is that there only can be countably many such sequences (Hrbacek and Jech 1999, p. 76). As different books might consist of different—but always finite—number of sentences, the set of all conceivable books is simply a union of many countably countable sets, and thus also has to be countable (Jech 2002, p. 50). This reasoning, slightly generalized, applies also to the general case of "all conceivable goods."[3] Therefore, contrary to what Murphy holds, the number of equations (prices) that the socialist planner should have to put on his list is definitely not uncountable. Even though our deliberations certainly lack mathematical rigor, it should be sufficiently clear that it is not so easy to give a real life example of an uncountable set. This should not be disturbing, however, since the crux of the matter is that the number of equations (prices) that the omnipotent planner has to come up with essentially does not matter, either in Lange's approach, or in reality.

Lange's Socialism and the Irrelevance of Murphy's Argument

Moving on to the second flaw we see in Murphy's argument, let us ask first why is it exactly that the price list published by the Planning Board should include not only present goods, but also future ones, i.e., those which *theoretically* could be produced at some later point in time? Obviously, Murphy believes that this is a precondition of the functioning of the market socialism model, as described first and foremost by Oskar Lange. It might be instructive, therefore, to briefly review the basics of Lange's so called competitive solution (Lange 1936, 1937).

The basic idea behind Lange's plan was to establish socialism right away, by abolishing capitalism and completely mimicking its pricing process (with the help of

[3] To see why, let us first observe that any good, g, can be thought of as a combination of certain factors of production. These factors have to be arranged somehow to yield the final good and let us call such an arrangement a production function and denote it by f. Since at any given moment in time there are only finitely many (at most countably many) atoms in the entire universe, there can only be finitely (at most countably) many factors of production. By definition, countable elements can be enumerated, so let us denote factors of production by a_i, where i belongs to the set of positive integers. Now, every expression $f(a_{i1}, a_{i2}, \ldots, a_{ik})$, where k is an integer denoting the number of factors of production used in the process, stands for a certain good g_f. Since we only have countably many factors of production and each production function uses only finitely many of them, we conclude—analogously to the case of books—that at any given moment in time only a countable number of goods could *theoretically* be produced. We may complicate the result by considering an infinite time horizon, but—as absurd as this sounds—the number of moments in eternity is also countable and even if for each moment we can come up with countably many goods, then the total number of goods in eternity is still countable (for it is a union of countably many countable sets).

a price list produced by the Planning Board). The best way to think of this is to imagine that on a certain day, say day n, we are in full capitalism, based on private ownership of the means of production, and one day later, on day $n+1$, a revolutionary communist shows up and imposes full socialism, that is collective ownership of the means of production. Thus writes Lange:

> Therefore, a comprehensive socialization programme can scarcely be achieved by gradual steps. A socialist government really intent upon socialism has to decide to carry out its socialization programme *at one stroke*, or to give it up altogether. (Lange 1937, p. 134)

Now, what exactly would change during this process, should the newly imposed socialist regime imitate capitalist calculation as proposed? First of all, the state would become the sole *owner*, but not the *controller* of the means of production. One of the main purposes of socialism, common to all socialists, is to abolish the hated phenomenon of interest—the dividend earned by capitalists by "merely" owning factors of production and renting them out. In case of Lange's model the state becomes the exclusive owner of all factors of production, and has the right to collect the whole interest (and then distribute it according to its own communist plans).[4]

Let us see further what happens on the day $n+1$ (we ignore the possibility of a revolution, civil war, etc., and abstract from the whole transition process). There are two crucial steps that need to be taken by the new government. First of all, the control of resources is to be decentralized, and second, the Central Planning Board is to publish a list of prices for all goods available in the economy. The first step requires that government signs contracts with previously expropriated capitalists, employing them as managers of plants, factories, and capital goods, which they had previously owned. They will still decide, to some extent, about the specific employment of factors of production, however—and this is of utmost importance—their income will be derived only from *managing*, not from owning capital goods. That is, if they sell a given capital good, they shall not be able to pocket the revenues, since their only legitimate income is labor income for managing (this income might be correlated in some way with the plant's overall performance).

The second step of publishing a list which should include all the factors of production available in the economy. The consumer market remains essentially "free"; i.e., people are allowed to spend their income on all goods and services offered at retail. With regard to the formation of prices, Lange refers to Fred Taylor's suggestion that under socialism, just as under capitalism, the price for any good

[4] In fact, as Lange notes, the assumption concerning full socialization, i.e., public ownership of all means of production, is merely a theoretical simplification and for a number of political reasons should not be put into practice (Lange 1936, p. 61). In a somewhat striking passage he even goes so far as to admit that (Lange 1937, p.135):

> [government] has to make it absolutely clear to everybody that socialism is not directed against private property as such, but only against that special type of private property which creates social privileges to the detriment of the great majority of the people or creates obstacles to economic progress, and that, consequently, all private property in the means of production and private enterprise which does have a *useful social function* will enjoy the full protection and support of the socialist State.

should equate quantity demanded and quantity supplied (Taylor 1929, pp. 6–8). If there should be a surplus (or a shortage) of a particular good, then the price would have to be lowered (or raised) in order to clear the market. What we have to bear in mind is that this model of pricing differs from capitalism in one significant way—only the *ultimate owner* of resources is changed. Market socialists would argue that it is a slight change, whereas the opponents, of course, would claim that it makes a great deal of a difference. Although the latter question is undoubtedly an interesting one, the present note is scarcely the place to ponder it. What remains important to us at the moment is the fact that the difference between Lange's market socialism and capitalism as defended by Mises is essentially *institutional* in nature. This is reflected in Lange's description of the socialist pricing process:

> Let the Central Planning Board start with a given set of prices chosen *at random*. All decisions of the managers of production and of the productive resources in public ownership and also all decisions of individuals as consumers and as suppliers of labor are made on the basis of these prices. As a result of these decisions the quantity demanded and supplied of each commodity is determined. If the quantity demanded of a commodity is not equal to the quantity supplied the price of that commodity has to be changed. It has to be raised if demand exceeds supply and lowered if the reverse is the case. Thus the Central Planning Board fixes a new set of prices which serves as a basis for new decisions, and which results in a new set of quantities demanded and supplied. Through this process of trial and error equilibrium prices are finally determined. Actually the process of trial and error would, of course, proceed on the basis of the prices *historically given*.... Thus accounting prices in a socialist economy can be determined by the same process of trial and error by which prices on a competitive market are determined (Lange 1936, p. 66).

The most important issue concerns, of course, the plan for organizing the capital market. The latter, however, does not lead to any kind of mathematical or accounting problem at all. In a capitalist economy all private businesses possess a finite amount of capital goods, and all of them were priced according to the capitalist scheme on day n. All the central planner has to do on the day $n+1$, is register all those capital goods along with their respective prices which were offered for them the day before (such a record could be general, published, say, on the Internet, but it could also be locally published on many decentralized lists).

Now, the public managers controlling plants and factories need only worry about the relationship between consumer goods they offer and prices for factors of production needed to produce them. Just as before, Lange's scheme works analogously to capitalism. The prices for factors of production, published on the central list, are influenced by the demand for those factors by the managers. And, of course, the managers' demand is derived from the amount of factors they need to produce certain goods and services.

To see this, suppose that suddenly consumers' demand for chairs increases. The price has to be raised in order to clear the market. This rise, however, leads to what Kirzner calls a "profit opportunity" (Kirzner 1992, p. 49), i.e., a spread between the (now) higher price for chairs, and prices for factors of production needed to produce

them (wood, etc.). Public managers (since their income should be based on the performance of their firms) increase their demand for wood and other materials, for it is profitable to increase the production of chairs. This leads to greater scarcity of wood and other materials, and so—in order to clear the market—the price has to be increased. In this way market socialism (with sufficiently "alert" managers) is supposed to imitate the imputation process present in the capitalist system.[5]

It should be clear that even though there are many problems with the market socialism model, the pricing process per se is not one of them. Of course, as we have just demonstrated, market socialism *can* produce prices; i.e., cardinal numbers enabling some sort of profit and loss calculations. The important thing is, however, that those prices will be of no use in guiding the central planner, since he himself produces the very guidance that he is supposed to follow. In stark contrast, prices under capitalism, though *technically* formed in a similar way, reflect private property constraints and the intellectual division of labor as fulfilled by many independently competitive decisions concerning the allocation and utilization of resources. When the entrepreneur is calculating under capitalism, he is in fact referring to the judgment of others and dealing with competing plans of other property owners. When the central planner is calculating under market socialism, he is only referring to his own opinion and judgment, because he has produced the numbers that he uses.[6] As Bornstein brilliantly puts it (1962, pp. 97–98):

> While prices can be set to equate supply with demand according to planner's preferences, these preferences cannot themselves be based on an independent calculation of opportunity costs, as reflected in independently determined scarcity prices, since the scarcity prices in use *are themselves fixed on the basis of planner's preferences.* (emphasis added)

The competitive solution, therefore, mimics the market and differs from capitalism only as far as the *ultimate ownership* of factors of production is concerned. This is why Murphy's argument that the list published by the Central Planning Board should include infinitely many (whether countably or uncountably) goods is incorrect. Murphy seems to think of socialism as like a computer program, which directs the allocation of resources. In such a program all the possible alternatives, i.e., all possible prices, should be somehow predefined (in particular, should equal 0 for as yet nonexistent goods), for otherwise the program would simply not know what to do once an innovation is introduced.

The problem is that neither socialism nor capitalism function like computer programs. Both are economic systems introduced and directed—one way or another—by human beings. Socialism may not provide an answer as to the profitability of introducing new goods, but it is a great misconception to infer from this that the central planners would not know what to do and how to price specific inventions (say a trip to Mars). According to Lange, they would simply set its price in relation to the prices of factors of production needed to produce it, and then correct it so that quantity supplied

[5] Incidentally, it is interesting to note that this decentralized market socialism might utilize "knowledge of particular place and time," since managers are deciding about the employment of factors.

[6] For a detailed treatment of problems inherent in the "competitive solution" see in particular Machaj (2007).

could equal quantity demanded. This method would not be significantly different than the one resorted to under capitalism.

Hence, just as private businesses do not need to publish the information in their catalogues about all the possible goods they could *theoretically* produce, so, too, the managers in socialism need not worry about them. Private businessmen can act efficiently without lists of all the possible consumer goods and services (i.e., all feasible combinations of factors of production), and similarly public managers under socialism can act without such lists. There is no difference between socialism and capitalism in this respect.

References

Anderson, Stephen R. 2004. *Doctor Dolittle's Delusion*. New Haven, Conn.: Yale University Press.

Andreski, Stanislav. 1972. *Social Sciences as Sorcery*. London: Andre Deutsch Ltd.

Aumann, Robert J. 1964. "Markets with a Continuum of Traders." *Econometrica* 32(1): 39–50.

Barone, Enrico. 1935. "Ministry of Production in a Collectivist State." In *Collectivist Economic Planning: Critical Studies on the Possibilities of Socialism*. Ed. F.A. Hayek. Clifton, N.J.: Augustus M. Kelley.

Bornstein, Morris. 1962. "The Soviet Price System." *American Economic Review* 52(1): 64–103.

Dickinson, H.D. 1933. "Price Formation in a Socialist Commonwealth." *The Economic Journal* 43(170): 237–50.

Hayek, F.A. 1935. Collectivist Economic Planning. London: George Routledge & Sons.

_____. 1940. "Socialist Calculation: The Completitive 'Solution'." *Economica* ns. 7(26):1 25–49.

Hrbacek, Karel, and Thomas Jech 1999. *Introduction to Set Theory*. New York: Marcel Dekker.

Jech, Thomas. 2002. *Set Theory*. Berlin, Heidelberg, New York: Springer Verlag.

Kirzner, Israel. 1992. *The Meaning of Market Process. Essays in the Development of Modern Austrian Economics*. London: Routledge.

Lange, Oskar. 1936. "On the Economic Theory of Socialism: Part One." *Review of Economic Studies* 4(1): 53–71.

_____. 1937. "On the Economic Theory of Socialism: Part Two." *Review of Economic Studies* 4(2): 123–142.

Machaj, Mateusz. 2007. "Market Socialism and the Property Problem." *Quarterly Journal of Austrian Economics* 10(4): 257–80.

Murphy, Robert P. 2006. "Cantor's Diagonal Argument: An Extension to the Socialist Calculation Debate." *Quarterly Journal of Austrian Economics* 9(2): 3–11.

Ormerod, Paul. 1995. *The Death of Economics*. New York: St. Martin's Press.

Sokal, Alan, and Jean Bricmont. 1999. *Fashionable Nonsense: Postmodern Intellectuals' Abuse of Science*. New York: Picador.

Taylor, Fred. 1929. "The Guidance of Production in a Socialist State." *American Economic Review* 19(1): 1–8.