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An Internal Weakness in Current Intertemporal Preference Theory

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ABSTRACT: This theoretical paper demonstrates that using utility functions to study intertemporal decisions creates an internal contradiction: nonfulfillment of comparability, which is a condition of constructing utility functions. It shows that Riemann's rearrangement theorem leads to basket incomparability for infinitely lived agents in discrete time and considers Austrian theory the best replacement for the model that implies this conflict. It suggests also that cardinality cannot fix this problem, since it does not eliminate the likelihood of generating conditionally convergent series unable to represent the rankings of potential consumption streams.

The most notable criticisms of current consumer theory are based on axioms contrary to its own: the Austrian school denies that satisfaction can be quantified at all, while behavioral economics questions the very possibility that a person possesses "rational preferences." Despite the fact that the first position is entirely a priori and the second entirely a posteriori, there is no doubt that both criticisms

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reach similar results when leveled not at consumer theory's axioms, but at consumer theory's corollaries. Examples include 1) the findings of Congdon, Kling, and Mullainathan (2011) on the higher cognitive cost and lower margin of error imposed by poverty, which fit into the Misesian fight against class polylogism; 2) the fact that the illusion of explanatory depth is a possible cause for the Hayekian fatal conceit; and 3) Monge's (2021) behavioral approach to the impossibility of economic calculation under socialism. Nevertheless, as Caldwell (1984, 126) points out: "A methodological critique of one system (no matter how perverse the system's tenets may seem) based wholly on the precepts of its rival (no matter how familiar those precepts may be) establishes nothing."

Does an authentic internal contradiction exist in modern consumer theory? Under its own assumptions, can current consumer theory be shown to be an inappropriate approach to explain consumer behavior? To answer these questions, it is necessary to analyze them in mathematical economics terms; and for the answer to be affirmative, a theoretical contradiction must be found following such a method. Therefore, this study will ignore behavioral economics (as its reasoning is basically empirical). Via reductio ad absurdum, this article will demonstrate that consumer theory applies utility functions inconsistently to the study of intertemporal decisions. It will also show that the Austrian school provides the best explanation of this theoretical failure and, furthermore, that it is current consumer theory's best replacement.[1]

WHAT IS RIEMANN'S REARRANGEMENT THEOREM?

Riemann's rearrangement theorem is an elegant but at the same time counterintuitive proposition that shows that in dealing with infinity, there are strange rules. It states that the terms of any infinite conditionally convergent series can be rearranged to converge to an arbitrary (finite) real number or diverge.

Simply put, a series is an expression of the idea of adding an infinite number of terms from a given starting point; that is, of a

[1] Perhaps the clearest precedent for this work is Murphy (2006), which uses Cantor's diagonal argument to demonstrate that the Langean solution to the problem of the theoretical impossibility of socialism leads to an uncountable set of equations.

summation of infinite summands. This notion is summarized by sigma notation: $a_0 + \cdots + a_N = \sum_{n=0}^{N} a_n$ summarizes the operation of summing all the terms a_n with $n \in \{0, \ldots, N\}$ and $N \in \mathbb{N}$ of the appropriate succession. If the natural number N tends to infinity, the sum becomes a series, so the formal representation of a series is given by $\lim_{N \to +\infty} \sum_{n=0}^{N} a_n \coloneqq \sum_{n=0}^{+\infty} a_n$.

Now, an infinite series $\sum_{n=0}^{+\infty} a_n$ is absolutely convergent when the sum of the absolute values of its terms is finite (that is, when $\sum_{n=0}^{+\infty} |a_n|$ is a finite number, a concrete value). On the other hand, an infinite series $\sum_{n=0}^{+\infty} b_n$ is conditionally convergent when $\sum_{n=0}^{+\infty} b_n$ is a finite number but $\sum_{n=0}^{+\infty} |b_n|$ is not. Hence, Riemann's rearrangement theorem essentially states that the order in which the terms are added does not change the result in the first type of infinite series but it does change the result in the second type of infinite series, whose terms can always be rearranged to obtain any value or diverge.

The following example is commonly used to expose this apparently inconceivable result. Consider the well-known Taylor-Maclaurin series expansion of ln(1-x):[2]

$$\ln(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n} \Rightarrow \ln(2) = -\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots .$$

This is known as the alternating harmonic series, $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ and ln(2) are convergent. However, $\sum_{n=1}^{+\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{+\infty} \frac{1}{n}$ diverges (it is the famous harmonic series: the sum of the inverse of all natural numbers except zero). To understand why, consider that

[2] The Taylor-Maclaurin series expansion of a one-dimensional function, $h(x)$, is the series $\sum_{n=0}^{+\infty} \frac{h^{(n)}(0)x^n}{n!}$, where $h^{(n)}(0)$ is the nth derivative of $h(x)$, valued at x=0, and $n!$ (n factorial), is equal to 1×2×…×(n-1)×n. At the risk of oversimplifying, Taylor's theorem states that every $h: \mathbb{R} \to \mathbb{R}$ function that is n times differentiable has a Taylor-Maclaurin representation; hence, $h(x) = \sum_{n=0}^{+\infty} \frac{h^{(n)}(0)x^n}{n!}$.

$$\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

Now, notice that $\frac{1}{3} > \frac{1}{4}, \frac{1}{5} > \frac{1}{8}, \frac{1}{6} > \frac{1}{8},$ and $\frac{1}{7} > \frac{1}{8}$ and compare each term in the series that is not a power of $\frac{1}{2}$ with the nearest power of $\frac{1}{2}$ below it. The resulting pattern shows that this chain of inequalities can continue indefinitely and the distance between powers of $\frac{1}{2}$ doubles with each term, so

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots >$$

$$1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \cdots$$

$$\Leftrightarrow$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

The right-hand side of this inequality is equal to one plus infinity times one-half $\left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = 1 + \frac{1}{2}\lim_{n\to\infty} n\right)$; ergo, it is infinite, and the left-hand side of the inequality, being bigger, is also infinite: therefore, $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges![3] This means that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$ is conditionally convergent, not absolutely convergent. Now, note that

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

[3] This very intuitive demonstration, the favorite of most modern calculus teachers, comes from the scholastic genius Nicolas Oresme! As this branch of mathematics developed, intuition became a general principle: if we have a series for which we can find a smaller and divergent series, the first series must be divergent; in formal terms, this is known as the direct comparison test.

$$\Rightarrow \frac{\ln(2)}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots$$

Summing these two expressions yields the following:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

$$+\frac{1}{2}\ln(2) = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots$$

$$\frac{3}{2}\ln(2) = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

Note that the resulting series has the same terms as the alternating harmonic series, but when the terms are rearranged, they lead to a different convergence value. Riemann's rearrangement theorem allows this series and any conditionally convergent series to converge to a different value. Now, it is time to analyze how this result affects mathematical economics.

INABILITY TO REPRESENT INTERTEMPORAL PREFERENCES

A clear example of the inadequacy of the neoclassical paradigm to deal with agents' intertemporal decisions is the study of "discrete time" itself: "difference equations" are modern consumer theory's basic tool for finding closed solutions. It can be shown that all difference equations that can be solved by recursive substitution can also be solved using the principle of mathematical induction on the subscripts of the variable of interest (although this is not the most efficient method). Hence, the prevailing analysis in modern economics reduces the complexity of praxeological time to that of the set of natural numbers! However, this section intends to disclose an internal contradiction in these methods: the assumed comparability of individuals' order of preferences might not be accomplished.

Consider an infinitely lived agent in a discrete time model with 1) exponential discounting, 2) additive separability, and 3) the same instantaneous utility function for consumption in each period. Taking a simple surjective, continuous, and strictly increasing function, $u: \mathbb{R}_+ \to \mathbb{R}$, as the instantaneous utility function yields at least one representation of this consumer's intertemporal utility function that does not preserve the order of his preferences. In this situation, there always exist baskets (consumption plans for each period) with positive utiles in some periods and negative utiles in others. However, because of *u*'s continuity and its codomain (where it is also surjective), it is clear that at least one consumption plan derived from the model leads to a conditionally convergent series. Simply put, any instantaneous utility of this type can become arbitrarily low, so in general, the possibility of an infinite number of periods in which instantaneous utility is negative (such that the total utility becomes a conditionally convergent series) cannot be ruled out.

Because of completeness of preferences,[4] that consumption plan is less desirable than some and preferred over others (or leaves the individual indifferent). But because of Riemann's rearrangement theorem, the conditionally convergent series of real numbers (utiles) can be rearranged so that it converges to another real number or diverges. When this happens, the preference relation reverses. So, the impossibility of reducing a person's decisions over an infinite period to a function is plausible. Nonetheless, this finding could also be interpreted as proposing the necessity of modeling only individuals who are myopic in the context of behavioral economics.

Note that invoking the rearrangement does not change the consumption period. It only permutes the order of the instantaneous utility summands. Each term's weighting (a discount factor power) is still the same and is particular to its corresponding period, so the basket (characterized by the time in which each quantity of consumption is enjoyed) is not changed. The order of the sum is being changed, not the order in which consumption occurred, but the resultant utility still changes.

This proposition could be extended—although with less intuitive force—to any static scenario where the utility function (dependent on infinite types of goods) could be transformed into a conditionally

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[4] Particularly comparability, the fundamental assumption of neoclassical economics that the individual can evaluate any basket.

convergent series. Harris Nover and Alan Hájek (2004) made a similar argument in a cardinal context in their study of the Pasadena game.[5]

However, it can still be questioned whether the solution to the problem of nonfulfillment of comparability implied by Riemann's rearrangement theorem is to consider intertemporal utility functions analytically (that is, by limiting the type of transformations), as in the case of the von Neumann–Morgenstern utility functions. The answer is partially no, because this would imply treating instantaneous utility functions like cardinal utility functions that admit positive linear transformations, which would still allow instantaneous utility functions to be multiplied (since the discount factor only needs to represent the relative rating between periods), making the addends arbitrarily large.[6] Additionally, a logarithmic instantaneous utility function, which takes values between negative and positive infinity for positive quantities of consumption and is one of the most used instantaneous utility functions, could be calibrated such that the present-discounted values of the time-stamped utilities become a conditionally convergent series.

The presence of this contradiction should indicate that the orthodox models' treatment of time is inappropriate because praxeological time is subjective and "the actor feels and experiences its course as he acts, . . . creating, discovering or, simply, realizing new ends and means" (Huerta de Soto 2005, 45–46). Furthermore, it is illusory to believe that most individuals cannot recognize, even tacitly, that time is a scarce resource: if the individual did not have infinite foresight, the series would not be conditionally convergent but rather would be a finite sum.

CONCLUSIONS

Orthodox consumer theory has internal contradictions. These are particularly visible in its analysis of intertemporal decision-making,

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[5] They created a game like the one used in the famous St. Petersburg paradox, but they modified the rewards so that the expected payoff was a conditionally convergent series; thus, they could make the game as unpleasant or attractive as they wanted.

[6] Furthermore, given that the structure of the intertemporal utility function would be the same as that of an expected utility function, all the observations of Harris and Hájek (2004) about the futility of imposing bounded cardinal utility functions to avoid inconsistencies in their Pasadena game would apply.

where the application of utility functions with unlimited foresight violates one of consumer theory's most decisive assumptions: comparability. Maybe these inconsistences do not make it worthy of being discarded. But these weaknesses reveal that human action is much more complex and intricate than what a function can capture, and at the same time much simpler and more instinctive than indifference relations as total orders. Utility functions may be elegant, but the ability to specify their domain makes them much more manipulable than actual human creativity.

Economic theory has patched up every hole that utility functions have opened: Arrow (1970) did so by demanding bounds for the cardinal utility functions, and Lowenstein (1992) did so too, by modeling the anomalies that standard intertemporal decisions and standard decisions under uncertainty shared. But what this article points out might be more alarming. It cannot be solved by restricting payments because the proposition rests on the utiles (not on units of consumption), and the type of utility function used cannot be restricted without invalidating most theoretical work to date. Both cannot be restricted because the proposition is general enough to hide which baskets will lead to a conditionally convergent series. This leaves only one option: to accept that if what the individual prefers cannot even be specified, then the use of utility functions is not appropriate for all scenarios—in short, to accept that the orthodox microeconomics of the last hundred years is not a general theory of human action.

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